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FORCED PLANE STRAIN MOTION of CYLINDRICAL SHELLS – A COMPARISON of SHELL THEORY with ELASTICITY THEORY

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by

PETER S. PAWLIK and HERBERT REISMANN

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Forced Plane Strain Motion of Cylindrical Shells--A Comparison of Shell Theory With Elasticity Theory

by

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NOMENCLATURE (I)

Dimensional Quantity	Physical Description	F-L-T Units
λ, μ	Lamé's constants	F/L^2
ρ	elastic material density	FT^2/L^4
$C_D = \sqrt{\frac{\lambda + 2\mu}{\rho}}$	dilatational wave speed	L/T
$C_s = \sqrt{\frac{\mu}{\rho}}$	shear wave speed	L/T
R	median surface radius	L
h	shell thickness	L
P_o	radial load intensity	F/L^2

NOMENCLATURE (II)

Dimensionless Quantity	To convert to dimensional form multiply by	Physical Description
$\gamma = C_s/C_D$	1	wave speed ratio
$\alpha = h/R$	1	thickness ratio
$1 - \frac{\alpha}{2} \leq r < 1 + \frac{\alpha}{2}$	R	radial coordinate
$-\pi \leq \theta \leq \pi$	1	plane polar angle
$0 < t < \infty$	$\frac{R}{C_D}$	time
w, v	$\frac{P_o R}{\rho C_D^2}$	radial and circumferential displacements respectively
$\tau_r, \tau_\theta, \tau$	P_o	radial, circumferential and shear stresses respectively
ϕ, [illegible]	$\frac{P_o R^2}{\rho C_D^2}$	displacement potentials

I. INTRODUCTION

The small motions of an isotropic, elastic medium, produced by a disturbance of its bounding surfaces, may be described mathematically by the equations of the three dimensional theory of elasticity together with appropriate boundary and initial conditions [1].(1) The first investigations of these equations of motion in cylindrical coordinates were conducted by Pochhammer [2] in 1876 and Chree [3] in 1889. Their studies dealt with the propagation of free harmonic waves in a solid cylinder which was infinite in extent in the direction of its generators. Since then several extensions and refinements of these initial studies have been made, most notably the addition of numerical data for the frequency equations. For a thorough discussion of Pochhammer's work and some of the subsequent investigations, the books by Love [4] and Kolsky [5] should be consulted.

The study of the motions of cylindrical shells using these equations is considerably more recent. It was only in the past two decades that an extensive effort was made to study the free harmonic vibrations of cylindrical shells as characterized by the three dimensional theory of elasticity. For a sample of the literature on this subject references [6] through [19] should be consulted. Also, recent studies have been made of the forced motion and transient response of cylindrical shells using this theory. For example, in 1964 Liu and Chang [20] investigated the transient radial displacement of an infinitely long cylindrical shell subjected to an internal axisymmetric blast load and

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(1)Numbers in brackets designate references at the end of the paper.

sudden temperature change. By applying the method of Mindlin and Goodman [21] to the problem, they were able to construct a solution in terms of the normal modes of vibration of the cylinder.

Subsequently Suzuki [22] considered the problem of a circular ring subjected to a transient pressure loading of both the inner and outer surfaces. He attacked the problem with a combination of Laplace Transforms for the time variable together with a harmonic analysis with respect to the angular coordinate in the plane of the ring. He thus formulated the general problem of a suddenly applied, exponentially decaying load, arbitrarily distributed over the lateral surfaces of the ring. However, he only presented solutions for the axially symmetric case.

In 1967 Garnet and Crouzet-Pascal [23] investigated the response of an infinite cylindrical shell imbedded in an infinite elastic medium produced by a plane dilatational wave traveling through the medium in a direction normal to the cylinder's axis. Their approach was to construct a train of incident pulses from steady-state components such that each pulse contained the time history of the transient stress in the incident wave. By making the time interval between successive pulses sufficiently large, the cylinder would return to its original, unstrained state before the arrival of the next pulse in the train. This occured because of the radiation of energy from the cylinder through the surrounding medium to infinity. This approach proved to be very successful and results were obtained to illustrate the time history of the stresses and displacements in the cylinder.

The examples cited above illustrate that the forced motion of a cylindrical shell as characterized by the three dimensional theory of elasticity is mathematically very complex. A quantitative description of the response is extremely difficult without the use of high speed computers. This is the principal reason for the long delay between the initial investigations of Pochhammer and Chree

and those just mentioned. The cylindrical shell, however, is a very common element with many and varied applications, therefore the need to analyze its dynamic response arose long before the means for carrying out such an analysis within the framework of the three dimensional theory of elasticity were available. This lead to the development of several, mathematically simpler, theories to describe the motion of cylindrical shells. These so called shell theories were based on the assumption that the radial thickness of the shell was much smaller than the radius of the median surface of the shell. With this assumption the dependent variables could be expanded into convergent power series in the thickness coordinate and the first one or two terms in these expansions would suffice to describe the response of the shell. A theory of this type was developed by Love [4] at about the same time as the Pochhammer and Chree investigations. Since then numerous other shell theories have been proposed. Most of these may be placed into one of the following three categories. The first type of theory is called a membrane theory. Here, no variation of the dependent variables through the thickness of the shell is permitted. Reference to this type of theory is made by Rayleigh [24] and the equations of motion for a cylindrical shell may be found in the books by Flügge [25] or Vlasov [26]. The second category contains the classical shell theories. These allow the dependent variables to vary linearly through the thickness of the shell but in such a manner that straight line elements normal to the median surface of the shell in the unstrained state remain normal during the motion of the shell. Furthermore, these elements retain their original length and contribute no rotatory inertia to the motion. Love's equations are contained in this category along with those of Flügge, Donnell, Vlasov and Sanders. The third category contains the improved theories. As in the classical theory the dependent variables are allowed to vary linearly through the thickness of

the shell. The improvement is obtained by allowing the previously mentioned line elements to rotate relative to the median surface and by including their rotatory inertia in the motion of the shell. Although these line elements are still required to remain straight and retain their original length, a further improvement is obtained by introducing a correction factor into the transverse shear force to compensate for this. The magnitude of the correction factor is obtained by matching the phase velocity of the lowest mode of propagation of free harmonic waves in the axial direction with that obtained from the three dimensional theory of elasticity [14]. The equations of motion of a cylindrical shell characterized by the improved theory may be found in the papers by Herrmann and Mirsky [27] and also Reismann and Medige [28].

As a result of the diversity of the proposed shell theories, the following question arises. For a cylindrical shell subjected to a specific disturbance, which of these theories predicts the response to within a given accuracy with the least effort? This question may be answered by comparing the response predicted by each of the shell theories to the response predicted by the three dimensional theory of elasticity for each specific disturbance. This approach, however, would negate the only advantage of the shell theories, which is their relative simplicity compared to the three dimensional theory of elasticity. An alternative approach is to carry out a comparison with the elasticity theory in only a few specific cases which represent the limits of the range of possible disturbances and shell geometries. From these few limiting cases rational estimates of the accuracy of the shell theories for various other disturbances and shell geometries could then be made. Some of these limiting cases have already been investigated and comparisons of the shell theories with elasticity theory have been made. For example Klosner in references [29] through [32] and Iyengar and Yogananda [33] have examined various problems involving

cylindrical shells which are statically loaded using elasticity theory and shell theories. However, no comparisons have as yet appeared for the other extreme case, that is where the load is suddenly applied. Since most real loading situations fall somewhere between these two extremes a comparison of the shell theories to elasticity theory for the latter case would be of great interest. Therefore, the primary purpose of this investigation will be to present such a comparison.

To accomplish this goal a cylindrical shell of circular cross section which is infinite in the direction of its generators is subjected to a suddenly applied force on its outer surface. This force is chosen to act only in the radial direction and also to be invariant in the direction of the generators of the cylinder. With these restrictions on the force the cylinder may be assumed to be in a state of plane strain and therefore only a plane section normal to its generators will be considered. The response of the shell predicted by the three dimensional theory of elasticity will be found using the method discussed by Suzuki [22]. Next, the response of the shell predicted by the Flügge Theory [25], a classical theory, and also the improved theory due to Herrmann and Mirsky [27] will be found from their corresponding Green's functions given by Pawlik and Reismann in [34].

The time history of the displacements and stresses at various points in the shell as characterized by the three theories will then be compared for a specific shell geometry and load distribution.

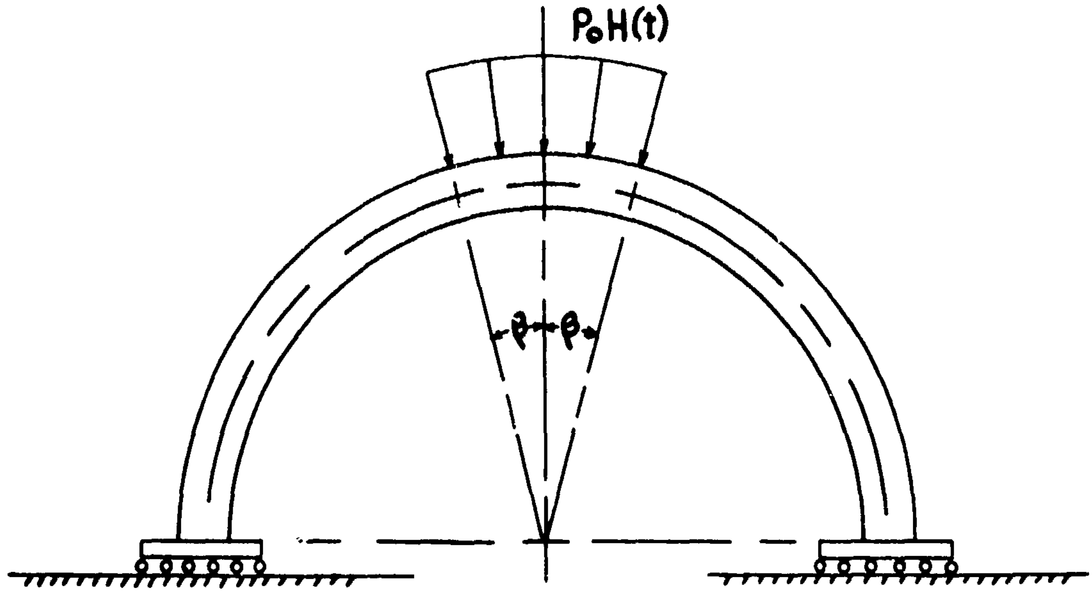

FIGURE 1-B

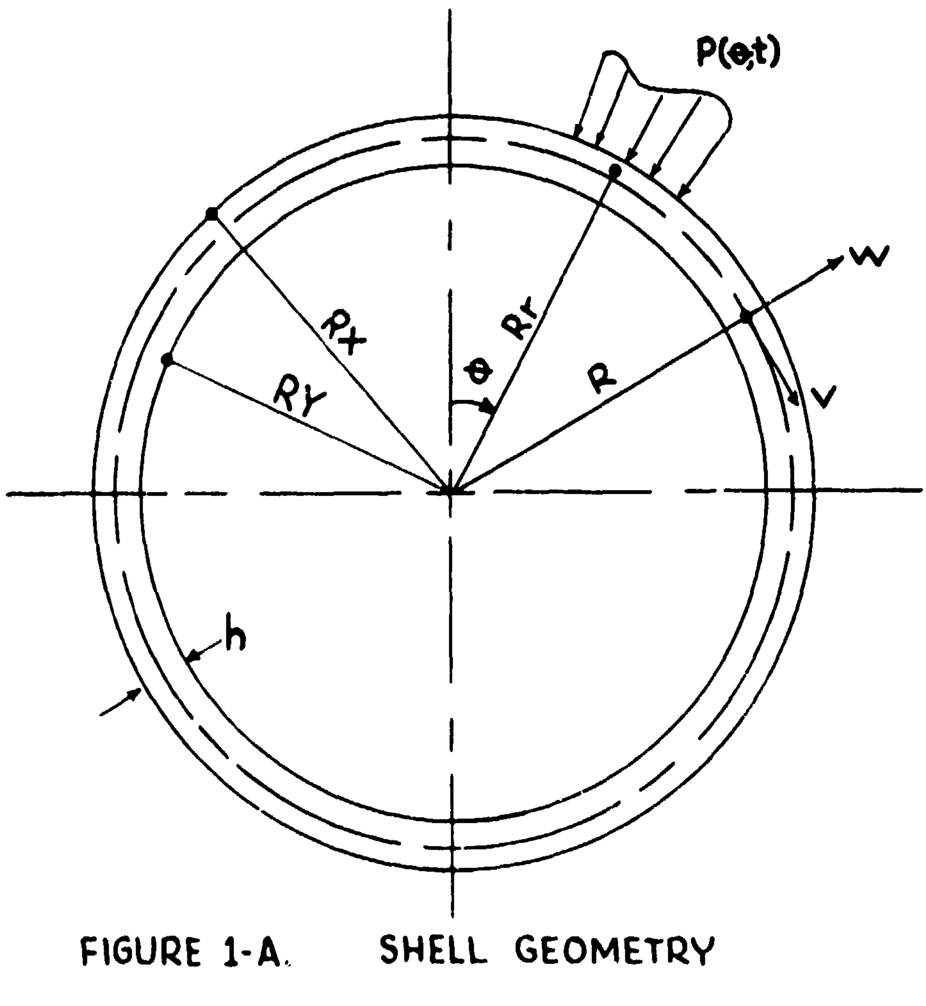

FIGURE 1-A. SHELL GEOMETRY

II. STATEMENT OF THE PROBLEM

An infinitely long, circular cylindrical shell, composed of an isotropic, linearly elastic material, is at rest and in an unstrained state. A load, directed radially inward, is suddenly applied to the outer cylindrical surface of the shell. The distribution of the load on this surface is constant along the generators of the cylinder but otherwise arbitrary.

To simplify the analysis of the resulting motion of the cylinder, its weight will be neglected. In this case it may be assumed that the motion of any plane section normal to the generators of the cylinder takes place entirely within that plane and is identical for all such sections. In other words the shell responds in a state of plane strain.

The initial geometry of a plane section of the shell together with a particular load is shown in Figure 1. The applied load may be represented symbolically as follows.

$$P(\theta, t) = P_0 \, g(\theta) \, H(t) \tag{1}$$

In (1), P_0 is a reference pressure, $g(\theta)$ describes the distribution of the load over the outer boundary of the shell and $H(t)$ is the Heaviside step function with respect to time:

$$H(t) = \begin{Bmatrix} 0 & , t < 0 \\ 1 & , t > 0 \end{Bmatrix} \tag{2}$$

The analysis of the motion for an arbitrary load distribution, $g(\theta)$, is facilitated by expressing $g(\theta)$ in terms of a Fourier series.

$$g(\theta) = \frac{1}{\pi}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta - b_n \sin n\theta)\right] \tag{3}$$

However, since ultimately a quantitative measure of the response is desired, the above series will necessarily be truncated after summing some finite number of terms (N). This results in an approximation of the actual distribution $g(\theta)$ by the finite sum $g_N(\theta)$. The accuracy of this approximation for any given value of N may be increased by applying the process of "smoothing," explained in Appendix I, to $g_N(\theta)$. In the present case this amounts to multiplying each term of the truncated series by the factor $\left(\frac{N}{n\pi} \sin \frac{n\pi}{N}\right)$.

$$g_N(\theta) = \frac{1}{\pi}\left[\frac{a_o}{2} + \sum_{n=1}^{N} (\alpha_n \cos n\theta - \beta_n \sin n\theta)\right] \tag{4}$$

where $\alpha_n = a_n \left(\frac{N}{n\pi} \sin \frac{n\pi}{N}\right)$

$$\beta_n = b_n \left(\frac{N}{n\pi} \sin \frac{n\pi}{N}\right)$$

The analysis will therefore be carried out for the load distribution $g_N(\theta)$ given by Equation (4).

All the variables used in the following analysis are in dimensionless form both for convenience and generality. The conversion to dimensional form is given in the Nomenclature.

III. THEORY OF ELASTICITY

A. Basic Equations

Navier's equations of motion for an isotropic, linearly elastic material may be written in dimensionless form as follows.

$$\gamma^2 \nabla^2 \vec{u} + (1 - \gamma^2) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) = \frac{\partial^2 \vec{u}}{\partial t^2} \tag{5}$$

Here $\vec{u}$ is the displacement vector, $\vec{\nabla}$ is the gradient operator, $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$ is the Laplician operator and γ is the wave speed ratio. Using Helmholtz's Theorem [36] the displacement vector field may be expressed as follows.

$$\vec{u} = \vec{\nabla} \varphi + \vec{\nabla} \times \vec{\psi} \quad , \quad \vec{\nabla} \cdot \vec{\psi} = 0 \tag{6}$$

φ is the scalar potential and $\vec{\psi}$ the vector potential of the vector field $\vec{u}$. Substituting (6) into (5) results in the following equation.

$$\vec{\nabla} \left[\nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial t^2} \right] + \vec{\nabla} \times \left[\gamma^2 \nabla^2 \vec{\psi} - \frac{\partial^2 \vec{\psi}}{\partial t^2} \right] = 0$$

This equation is satisfied if

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial t^2}$$

and (7)

$$\gamma^2 \nabla^2 \vec{\psi} = \frac{\partial^2 \vec{\psi}}{\partial t^2} .$$

For the case of plane strain in cylindrical coordinates the displacement vector and potentials are

$$\begin{aligned} \vec{u} &= w(r, \theta, t)\, \hat{e}_r + v(r, \theta, t)\, \hat{e}_\theta \\ \varphi &= \varphi(r, \theta, t) \\ \vec{\psi} &= \psi(r, \theta, t)\, \hat{e}_z \end{aligned} \tag{8}$$

where $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_z$ are the unit vectors in the radial, circumferential and axial directions respectively. The kinematic and constitutive relations for the plane strain case are

$$
\begin{aligned}
e_r &= \frac{\partial w}{\partial r} \\
e_\theta &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r} \\
s &= \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\tau_r &= e_r + (1 - 2\gamma^2)e_\theta \\
\tau_\theta &= e_\theta + (1 - 2\gamma^2)e_r \\
\tau &= \gamma^2 s
\end{aligned} \tag{10}
$$

where e_r, e_θ, τ_r, τ_θ are the normal strains and corresponding stresses while s and τ are the shear angle and shear stress respectively.

Substitution of (8) into (6) and (7) yields the following set of equations:

$$
\begin{aligned}
\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\varphi &= \frac{\partial^2 \varphi}{\partial t^2} \\
\gamma^2\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\psi &= \frac{\partial^2 \psi}{\partial t^2}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
w &= \frac{\partial \varphi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\
v &= \frac{1}{r}\frac{\partial \varphi}{\partial \theta} - \frac{\partial \psi}{\partial r}
\end{aligned} \tag{12}
$$

Substituting (12) into (9), then (9) into (10) results, with the aid of (11), in the following relations:

$$
\begin{aligned}
\tau_r &= \frac{\partial^2 \varphi}{\partial t^2} - 2\gamma^2\left[\frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\right] \\
\tau_\theta &= \frac{\partial^2 \varphi}{\partial t^2} - 2\gamma^2\left[\frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\right]
\end{aligned} \tag{13}
$$

$$\tau = \frac{\partial^2 \psi}{\partial t^2} - 2\gamma^2 \left[\frac{\partial^2 \psi}{\partial r^2} - \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right) \right] \tag{13}$$

The initial conditions for the problem under consideration are

$$\begin{aligned} &\varphi(r,\theta,0) = \psi(r,\theta,0) = 0 \\ &\left.\frac{\partial \varphi}{\partial t}\right|_{t=0} = \left.\frac{\partial \psi}{\partial t}\right|_{t=0} = 0 \end{aligned} \tag{14}$$

The boundary conditions are

$$\begin{aligned} &\sigma_r(X,\theta,t) = - g_N(\theta)H(t) \\ &\tau(X,\theta,t) = 0 \\ &\sigma_r(Y,\theta,t) = 0 \\ &\tau(Y,\theta,t) = 0 \end{aligned} \tag{15}$$

where $X = x = 1 + \frac{\alpha}{2}$ and $Y = y = 1 - \frac{\alpha}{2}$ are the outer and inner radii of the shell, respectively.

B. Solution of the Equations of Elasticity

The solution of the system of equations, boundary conditions and initial conditions (11) through (15) will now be obtained by a combination of harmonic analysis and Laplace transformation.

In view of the boundary conditions (15) and the form of $g_N(\theta)$ given by (4), the dependent variables in (11) through (15) may be assumed to be in the following form.

$$\phi(r,\theta,t) = -\frac{1}{\pi}\left[\frac{a_o}{2}\Phi_o(r,t) + \sum_{n=1}^{N}\Phi_n(r,t)(\alpha_n\cos n\theta - \beta_n\sin n\theta)\right] \quad (16\text{-a})$$

$$\psi(r,\theta,t) = \frac{1}{\pi}\sum_{n=1}^{N}\Psi_n(r,t)(\alpha_n\sin n\theta + \beta_n\cos n\theta) \quad (16\text{-b})$$

$$w(r,\theta,t) = \frac{1}{\pi}\left[\frac{a_o}{2}W_o(r,t) + \sum_{n=1}^{N}W_n(r,t)(\alpha_n\cos n\theta - \beta_n\sin n\theta)\right] \quad (17\text{-a})$$

$$v(r,\theta,t) = \frac{1}{\pi}\sum_{n=1}^{N}V_n(r,t)(\alpha_n\sin n\theta + \beta_n\cos n\theta) \quad (17\text{-b})$$

$$\tau_r(r,\theta,t) = -\frac{1}{\pi}\left[\frac{a_o}{2}S_o^r(r,t) + \sum_{n=1}^{N}S_n^r(r,t)(\alpha_n\cos n\theta - \beta_n\sin n\theta)\right] \quad (18\text{-a})$$

$$\tau_\theta(r,\theta,t) = -\frac{1}{\pi}\left[\frac{a_o}{2}S_o^\theta(r,t) + \sum_{n=1}^{N}S_n^\theta(r,t)(\alpha_n\cos n\theta - \beta_n\sin n\theta)\right] \quad (18\text{-b})$$

$$\tau(r,\theta,t) = \frac{1}{\pi}\sum_{n=1}^{N}T_n(r,t)(\alpha_n\sin n\theta + \beta_n\cos n\theta) \quad (18\text{-c})$$

The modal coefficients Φ_n, Ψ_n etc. are determined by substituting the assumed solutions (16) through (18) into the relations (11) through (15). This

results in the following set of equations.

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right]\Phi_n = \frac{\partial^2 \Phi_n}{\partial t^2} \tag{19-a}$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right]\Psi_n = \frac{1}{\gamma^2}\frac{\partial^2 \Psi_n}{\partial t^2} \tag{19-b}$$

$$\Psi_o(r,t) \equiv 0 \tag{19-c}$$

$$\Phi_n(r,0) = \Psi_n(r,0) = 0 \tag{20-a}$$

$$\left.\frac{\partial \Phi_n}{\partial t}\right|_{t=0} = \left.\frac{\partial \Psi_n}{\partial t}\right|_{t=0} = 0 \tag{20-b}$$

$$S_n^r(X,t) = H(t) \quad , \quad S_n^r(Y,t) = 0 \tag{21-a}$$

$$T_n(X,t) = 0 \quad , \quad T_n(Y,t) = 0 \tag{21-b}$$

$$W_n(r,t) = \frac{\partial \Phi_n}{\partial r} + \frac{n}{r}\Psi_n \tag{22-a}$$

$$V_n(r,t) = \frac{\partial \Psi_n}{\partial r} + \frac{n}{r}\Phi_n \tag{22-b}$$

$$S_n^r(r,t) = \frac{\partial^2 \Phi_n}{\partial t^2} - \frac{2\gamma^2}{r}\left[W_n + nV_n\right] \tag{22-c}$$

$$T_n(r,t) = \frac{\partial^2 \Psi_n}{\partial t^2} - \frac{2\gamma^2}{r}\left[V_n - nW_n\right] \tag{22-d}$$

$$S_n^{\theta}(r,t) = (1 - 2\gamma^2)\frac{\partial^2 \Phi_n}{\partial t^2} + \frac{2\gamma^2}{r}\left[W_n - nV_n\right] \tag{22-e}$$

where $n = \{0, 1, 2, \ldots, N\}$.

The solution of the differential equations (19) may be obtained by application of the Laplace transformation defined below.

$$\bar{f}(r,P) = \int_0^{\infty} f(r,t)e^{-Pt}dt \tag{23-a}$$

$$f(r,t) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \bar{f}(r,P)e^{Pt}dP \tag{23-b}$$

In the above integral, known as Bromwich's integral formula, $i = \sqrt{-1}$ and the real number C is chosen so that $P = C$ lies to the right of all the singularities of $\bar{f}(r,P)$ in the complex P-plane.

Applying the transformation (23-a) to the set of equations (19) through (22) results in the following system of equations:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(P^2 + \frac{n^2}{r^2}\right)\right]\bar{\Phi}_n = 0 \tag{24-a}$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{P^2}{\gamma^2} + \frac{n^2}{r^2}\right)\right]\bar{\Psi}_n = 0 \tag{24-b}$$

$$\bar{\Psi}_o(r,P) \equiv 0 \tag{24-c}$$

$$\bar{S}^r_n(X,P) = \frac{1}{P}, \quad \bar{S}^r_n(Y,P) = 0 \tag{25-a}$$

$$\bar{T}_n(X,P) = 0, \quad \bar{T}_n(Y,P) = 0 \tag{25-b}$$

$$\bar{W}_n(r,P) = \frac{d\bar{\Phi}_n}{dr} + \frac{n}{r}\bar{\Psi}_n \tag{26-a}$$

$$\bar{V}_n(r,P) = \frac{d\bar{\Psi}_n}{dr} + \frac{n}{r}\bar{\Phi}_n \tag{26-b}$$

$$\bar{S}^r_n(r,P) = P^2\bar{\Phi}_n - \frac{2\gamma^2}{r}[\bar{W}_n - n\bar{V}_n] \tag{26-c}$$

$$\bar{T}_n(r,P) = P^2\bar{\Psi}_n - \frac{2\gamma^2}{r}[\bar{V}_n - n\bar{W}_n] \tag{26-d}$$

$$\bar{S}^\theta_n(r,P) = (1 - 2\gamma^2)P^2\bar{\Phi}_n + \frac{2\gamma^2}{r}[\bar{W}_n - n\bar{V}_n] \tag{26-e}$$

where $n = \{0,1,2, \ldots ,N\}$.

The solution of equations (24) is (see for example [35])

$$\bar{\Phi}_n(r,P) = A_n(P)I_n(Pr) + B_n(P)K_n(Pr) \tag{27-a}$$

$$\bar{\Psi}_n(r,P) = C_n(P)I_n\left(\frac{Pr}{\gamma}\right) + D_n(P)K_n\left(\frac{Pr}{\gamma}\right) \tag{27-b}$$

$$\bar{\Psi}_0(r,P) \equiv 0 \tag{27-c}$$

where I_n and K_n are the modified Bessel functions of the first and second kind respectively. The constants A_n, B_n, C_D and D_n are determined as follows. Substitute (27) into the relations (26) to obtain expressions for the modal stresses. Next, substitute these expressions into the boundary conditions (25). This results in a set of linear algebraic equations from which the constants A_n, B_n, C_n and D_n may be uniquely determined. After substituting the constants thus obtained into (27) it is observed that all the modified Bessel functions may be conveniently grouped into four new functions called the cross products of the modified Bessel functions. These are defined below:

$$\bar{F}_n^{(1)}(P,X,Y) \equiv I_n(PX)K_n(PY) - I_n(PY)K_n(PX) \tag{28-a}$$

$$\bar{F}_n^{(2)}(P,X,Y) \equiv PY[I_n(PX)K_n'(PY) - I_n'(PY)K_n(PX)] \tag{28-b}$$

$$\bar{F}_n^{(3)}(P,X,Y) \equiv PX[I_n'(PX)K_n(PY) - I_n(PY)K_n'(PX)] \tag{28-c}$$

$$\bar{F}_n^{(4)}(P,X,Y) \equiv \begin{cases} I_0'(PX)K_0'(PY) - I_0'(PY)K_0'(PX) & , n = 0 \\ P^2XY[I_n'(PX)K_n'(PY) - I_n'(PY)K_n'(PX)] & , n > 0 \end{cases} \tag{28-d}$$

where a prime denotes differentiation with respect to the argument of the function. The properties of these functions used in the forthcoming analysis are listed in Appendix II.

In terms of these cross products the solution may now be written as

$$\bar{\Phi}_n(r,P) = \frac{\bar{C}_n^{(\varphi)}(r,P)}{P^3\bar{D}_n(P)} \tag{29-a}$$

$$\bar{\Psi}_n(r,P) = -\frac{\bar{C}_n^{(\psi)}(r,P)}{P^3\bar{D}_n(P)} \quad ; \quad \bar{\Psi}_0 \equiv 0 \tag{29-b}$$

where $\bar{C}_n^{(\varphi)}$ and $\bar{C}_n^{(\psi)}$ are given below.

For $n = 0$:

$$\bar{C}_o^{\varphi}(r,P) = P^2\bar{F}_o^{(1)}(P,r,Y) - \frac{2\gamma^2}{Y^2}\bar{F}_o^{(2)}(P,r,Y) \tag{30-a}$$

$$\bar{D}_o(P) = P^2\bar{F}_o^{(1)}(P,X,Y) - \frac{2\gamma^2}{Y^2}\bar{F}_o^{(2)}(P,X,Y)$$

$$- \frac{2\gamma^2}{Y^2}\bar{F}_o^{(3)}(P,X,Y) + \frac{4\gamma^4}{XY}\bar{F}_o^{(4)}(P,X,Y) \tag{30-b}$$

For n = 1:

$$\bar{C}_1^{\varphi}(r,P) = \frac{1}{P^2}\bar{A}_1^{\varphi}(r,P) \tag{31-a}$$

$$\bar{C}_1^{\psi}(r,P) = \frac{1}{P^2}\bar{A}_1^{\psi}(r,P) \tag{31-b}$$

$$\bar{D}_1(P) = \frac{1}{P^2}\bar{A}_1^{\Delta}(P) \tag{31-c}$$

For $n = \{2,3,4,\ldots,N\}$:

$$\bar{C}_n^{\varphi}(r,P) = \bar{A}_n^{\varphi}(r,P) + (n^2-1)\bar{B}_n^{\varphi}(r,P) \tag{32-a}$$

$$\bar{C}_n^{\psi}(r,P) = \bar{A}_n^{\psi}(r,P) + (n^2-1)\bar{B}_n^{\psi}(r,P) \tag{32-b}$$

$$\bar{D}_n(P) = \bar{A}_n^{\Delta}(P) + (n^2-1)\bar{B}_n^{\Delta}(P) \tag{32-c}$$

The functions $\bar{A}_n^{\varphi}$ etc. are defined below in terms of the cross products for $n = \{1,2,3,\ldots,N\}$.

$$\bar{A}_n^{\Delta}(P) = \frac{8\gamma^4 n^2}{X^2Y^2} + \left(P^2 + \frac{4\gamma^2 n^2}{X^2}\right)\left(P^2 + \frac{4\gamma^2 n^2}{Y^2}\right)\bar{F}_n^{(1,1)}(P)$$

$$- \frac{4\gamma^2}{Y^2}\left(P^2 + \frac{4\gamma^2 n^2}{X^2}\right)\bar{F}_n^{(1,2)}(P) - \frac{4\gamma^2}{X^2}\left(P^2 + \frac{4\gamma^2 n^2}{Y^2}\right)\bar{F}_n^{(1,3)}(P)$$

$$+ \frac{8\gamma^4}{X^2Y^2}\left[\bar{F}_n^{(1,4)}(P) + \bar{F}_n^{(2,3)}(P)\right]$$

$$\bar{B}_n^{\Delta}(P) = \frac{4\gamma^4}{P^4X^2Y^2}\left\{\frac{8\gamma^4 n^2(n^2-1)}{X^2Y^2} + 4\gamma^2 n^2 P^2\left(\frac{1}{Y^2} + \frac{1}{X^2}\right)\right.$$

$$\left. + n^2\left[\frac{4\gamma^4 n^2(n^2-1)}{X^2Y^2} + 4\gamma^2 n^2 P^2\left(\frac{1}{Y^2} + \frac{1}{X^2}\right) + P^4\left(\frac{X^2}{Y^2} + \frac{Y^2}{X^2}\right)\right]\bar{F}_n^{(1,1)}(P)\right.$$

(continued on next page)

$$-\frac{X^2}{Y^2}\left[\frac{4\gamma^4 n^2(n^2-1)}{X^4} + \frac{4\gamma^2 n^2 P^2}{X^2} + P^4\right]\bar{F}_n^{(2,2)}(P)$$

$$-\frac{Y^2}{X^2}\left[\frac{4\gamma^4 n^2(n^2-1)}{Y^4} + \frac{4\gamma^2 n^2 P^2}{Y^2} + P^4\right]\bar{F}_n^{(3,3)}(P)$$

$$+\frac{4\gamma^4(n^2-1)}{X^2Y^2}\bar{F}_n^{(4,4)}(P) - \frac{4\gamma^2 n^2 P^2}{X^2}\bar{F}_n^{(1,2)}(P)$$

$$\left.-\frac{4\gamma^2 n^2 P^2}{Y^2}\bar{F}_n^{(1,3)}(P) + 4\gamma^2 P^2\left[\frac{1}{X^2}\bar{F}_n^{(3,4)}(P) + \frac{1}{Y^2}\bar{F}_n^{(2,4)}(P)\right.\right\}$$

where $\bar{F}_n^{(i,j)}(P) \equiv \frac{1}{2}\left[\bar{F}_n^{(i)}(P,X,Y)\bar{F}_n^{(j)}\left(\frac{P}{\gamma},X,Y\right) + \bar{F}_n^{(i)}\left(\frac{P}{\gamma},X,Y\right)\bar{F}_n^{(j)}(P,X,Y)\right]$

$$\bar{A}_n^{\varphi}(r,P) \equiv -\frac{4\gamma^4 n^2}{X^2Y^2}\left[\bar{F}_n^{(1)}(P,X,r) - \bar{F}_n^{(3)}(P,X,r)\right]$$

$$+\left(P^2 + \frac{2\gamma^2 n^2}{X^2}\right)\left(P^2 + \frac{4\gamma^2 n^2}{Y^2}\right)\bar{F}_n^{(1)}(P,r,Y)\bar{F}_n^{(1)}\left(\frac{P}{\gamma},X,Y\right)$$

$$-\frac{2\gamma^2}{Y^2}\left(P^2 + \frac{2\gamma^2 n^2}{X^2}\right)\left[\bar{F}_n^{(1)}(P,r,Y)\bar{F}_n^{(2)}\left(\frac{P}{\gamma},X,Y\right)\right.$$

$$\left.+\bar{F}_n^{(2)}(P,r,Y)\bar{F}_n^{(1)}\left(\frac{P}{\gamma},X,Y\right)\right] - \frac{2\gamma^2}{X^2}\left(P^2 + \frac{4\gamma^2 n^2}{Y^2}\right)\bar{F}_n^{(1)}(P,r,Y)\bar{F}_n^{(3)}\left(\frac{P}{\gamma},X,Y\right)$$

$$+\frac{4\gamma^4}{X^2Y^2}\left[\bar{F}_n^{(1)}(P,r,Y)\bar{F}_n^{(4)}\left(\frac{P}{\gamma},X,Y\right) + \bar{F}_n^{(2)}(P,r,Y)\bar{F}_n^{(3)}\left(\frac{P}{\gamma},X,Y\right)\right]$$

$$\bar{A}_n^{\psi}(r,P) \equiv -\frac{2\gamma^2 n}{Y^2}\left[\left(P^2 + \frac{2\gamma^2 n^2}{X^2}\right)\bar{F}_n^{(1)}\left(\frac{P}{\gamma},X,r\right) - \frac{2\gamma^2}{X^2}\bar{F}_n^{(3)}\left(\frac{P}{\gamma},X,r\right)\right]$$

$$+\frac{2\gamma^2 n}{X^2}\left(P^2 + \frac{4\gamma^2 n^2}{Y^2}\right)\bar{F}_n^{(1)}\left(\frac{P}{\gamma},r,Y\right)\bar{F}_n^{(1)}(P,X,Y)$$

$$-\frac{4\gamma^4 n}{X^2Y^2}\left[\bar{F}_n^{(1)}\left(\frac{P}{\gamma},r,Y\right)\bar{F}_n^{(2)}(P,X,Y) + \bar{F}_n^{(2)}\left(\frac{P}{\gamma},r,Y\right)\bar{F}_n^{(1)}(P,X,Y)\right]$$

$$-\frac{2\gamma^2 n}{X^2}\left(P^2 + \frac{4\gamma^2 n^2}{Y^2}\right)\bar{F}_n^{(1)}\left(\frac{P}{\gamma},r,Y\right)\bar{F}_n^{(3)}(P,X,Y)$$

$$+\frac{4\gamma^4 n}{X^2Y^2}\left[\bar{F}_n^{(1)}\left(\frac{P}{\gamma},r,Y\right)\bar{F}_n^{(4)}(P,X,Y)+\bar{F}_n^{(2)}\left(\frac{P}{\gamma},r,Y\right)\bar{F}_n^{(3)}(P,X,Y)\right]$$

$$\bar{B}_n^{\varphi}(r,P)\equiv\frac{4\gamma^4}{P^2X^2Y^4}\left\{2\gamma^2n^2\left[\bar{F}_n^{(3)}(P,X,r)-\bar{F}_n^{(1)}(P,X,r)\right]\right.$$

$$+n^2(P^2X^2+2\gamma^2n^2)\bar{F}_n^{(1)}(P,r,Y)\bar{F}_n^{(1)}\left(\frac{P}{\gamma},X,Y\right)$$

$$-2\gamma^2n^2\bar{F}_n^{(1)}(P,r,Y)\bar{F}_n^{(3)}\left(\frac{P}{\gamma},X,Y\right)+2\gamma^2\bar{F}_n^{(2)}(P,r,Y)\bar{F}_n^{(4)}\left(\frac{P}{\gamma},X,Y\right)$$

$$\left.-(P^2X^2+2\gamma^2n^2)\bar{F}_n^{(2)}(P,r,Y)\bar{F}_n^{(2)}\left(\frac{P}{\gamma},X,Y\right)\right\}$$

$$\bar{B}_n^{\psi}(r,P)\equiv\frac{4\gamma^4 n}{P^2X^2Y^4}\left\{2\gamma^2\bar{F}_n^{(3)}\left(\frac{P}{\gamma},X,r\right)-\left(P^2X^2+2\gamma^2n^2\right)\bar{F}_n^{(1)}\left(\frac{P}{\gamma},X,r\right)\right.$$

$$+2\gamma^2n^2\bar{F}_n^{(1)}\left(\frac{P}{\gamma},r,Y\right)\bar{F}_n^{(1)}(P,X,Y)-2\gamma^2n^2\bar{F}_n^{(1)}\left(\frac{P}{\gamma},r,Y\right)\bar{F}_n^{(3)}(P,X,Y)$$

$$\left.-2\gamma^2\bar{F}_n^{(2)}\left(\frac{P}{\gamma},r,Y\right)\bar{F}_n^{(2)}(P,X,Y)+2\gamma^2\bar{F}_n^{(2)}\left(\frac{P}{\gamma},r,Y\right)\bar{F}_n^{(4)}(P,X,Y)\right\}$$

The above relations will be referred to as equations (33).

The solution in the time domain will now be obtained by substituting (29) into the Bromwich integral formula (23-b).

$$\Phi_n(r,t)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{\bar{C}_n^{\varphi}(r,P)}{P^3\bar{D}_n(P)}e^{Pt}dP \tag{34-a}$$

$$\Psi_n(r,t)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{\bar{C}_n^{\psi}(r,P)}{P^3\bar{D}_n(P)}e^{Pt}dP \tag{34-b}$$

For $r>0$, each of the functions $\bar{C}_n^{\varphi}(r,P)$, $\bar{C}_n^{\psi}(r,P)$ and $\bar{D}_n(P)$ is (a) an entire function of P, (b) symmetrical in P and (c) nonzero at $P = 0$. $\bar{D}_n(P)$ has a denumerable infinity of simple zeroes located along the imaginary axis of the complex P-plane for each value of n. If the magnitude of these zeroes is denoted by ω_{nj} then

$$\bar{D}_n(\pm i\,\omega_{nj}) = 0 \quad ; \quad j = 0,1,2,\ldots \tag{35-a}$$

with

$$0 < \omega_{no} < \omega_{n1} < \omega_{n2} < \cdots \tag{35-b}$$

The integrals (34-a, b) will now be evaluated by contour integration. From the above discussion we conclude that both integrands have simple poles at $P = \pm i\,\omega_{nj}$ and a pole of order three at the origin. By applying the residue theorem to the integral around a Bromwich contour with $C > 0$ the solutions are obtained in the following form.

For $t > 0$:

$$\Phi_n(r,t) = \Phi_n^{(S)}(r,t) + \sum_{j=0}^{\infty} C_n^{\varphi}(r, \omega_{nj}) Q_{nj}(t) \tag{36-a}$$

$$\Psi_n(r,t) = \Psi_n^{(S)}(r,t) + \sum_{j=0}^{\infty} C_n^{\psi}(r, \omega_{nj}) Q_{nj}(t) \tag{36-b}$$

where

$$\Phi_n^{(S)}(r,t) = \frac{1}{2}\left[\frac{\partial^2}{\partial P^2}\left(\frac{\bar{C}_n^{\varphi}(r,P)e^{Pt}}{\bar{D}_n(P)}\right)\right]_{P=0} \tag{37-a}$$

$$\Psi_n^{(S)}(r,t) = \frac{1}{2}\left[\frac{\partial^2}{\partial P^2}\left(\frac{\bar{C}_n^{\psi}(r,P)e^{Pt}}{\bar{D}_n(P)}\right)\right]_{P=0} \tag{37-b}$$

$$C_n^{\varphi}(r, \omega) = \bar{C}_n^{\varphi}(r, \pm i\omega) \tag{38-a}$$

$$C_n^{\psi}(r, \omega) = \bar{C}_n^{\psi}(r, \pm i\omega) \tag{38-b}$$

$$D_n(\omega) = \bar{D}_n(\pm i\omega) \tag{38-c}$$

$$Q_{nj}(t) = \frac{\cos \omega_{nj} t}{\omega_{nj}^4 G_n(\omega_{nj})} \tag{39-a}$$

$$G_n(\omega) = \bar{G}_n(\pm i\omega) \tag{39-b}$$

$$\bar{G}_n(P) = \frac{1}{2P}\frac{d\bar{D}_n(P)}{dP} \tag{39-c}$$

By substituting the potentials (36) into (32) we obtain the desired solutions for the modal displacements and stresses.

$$W_n(r,t) = W_n^{(s)}(r,t) + \sum_{j=0}^{\infty} W_{nj}(r)Q_{nj}(t) \tag{40-a}$$

$$V_n(r,t) = V_n^{(s)}(r,t) + \sum_{j=0}^{\infty} V_{nj}(r)Q_{nj}(t) \tag{40-b}$$

$$S_n^r(r,t) = S_n^{r(s)}(r,t) + \sum_{j=0}^{\infty} S_{nj}^r(r)Q_{nj}(t) \tag{40-c}$$

$$S_n^\theta(r,t) = S_n^{\theta(s)}(r,t) + \sum_{j=0}^{\infty} S_{nj}^\theta(r)Q_{nj}(t) \tag{40-d}$$

$$T_n(r,t) = T_n^{(s)}(r,t) + \sum_{j=0}^{\infty} T_{nj}(r)Q_{nj}(t) \tag{40-e}$$

where

$$W_{nj}(r) = \frac{\partial}{\partial r} C_n^{\varphi}(r, \omega_{nj}) + \frac{n}{r} C_n^{\psi}(r, \omega_{nj}) \tag{41-a}$$

$$V_{nj}(r) = \frac{\partial}{\partial r} C_n^{\psi}(r, \omega_{nj}) + \frac{n}{r} C_n^{\varphi}(r, \omega_{nj}) \tag{41-b}$$

$$S_{nj}^r(r) = - \omega_{nj}^2 C_n^{\varphi}(r, \omega_{nj}) - \frac{2\gamma^2}{r}[W_{nj} - nV_{nj}] \tag{41-c}$$

$$S_{nj}^\theta(r) = - \omega_{nj}^2 (1 - 2\gamma^2) C_n^{\varphi}(r, \omega_{nj}) + \frac{2\gamma^2}{r}[W_{nj} - nV_{nj}] \tag{41-d}$$

$$T_{nj}(r) = - \omega_{nj}^2 C_n^{\psi}(r, \omega_{nj}) - \frac{2\gamma^2}{r}[V_{nj} - nW_{nj}] \tag{41-e}$$

The functions $C_n^{\varphi}(r, \omega)$, $C_n^{\psi}(r, \omega)$ etc. may be conveniently expressed in terms of the cross products of the Bessel functions defined below.

$$F_n^{(1)}(\omega, x, y) = J_n(\omega x)Y_n(\omega y) - J_n(\omega y)Y_n(\omega x) \tag{42-a}$$

$$F_n^{(2)}(\omega, x, y) = \omega y[J_n(\omega x)Y_n'(\omega y) - J_n'(\omega y)Y_n(\omega x)] \tag{42-b}$$

$$F_n^{(3)}(\omega, x, y) = \omega x[J_n'(\omega x)Y_n(\omega y) - J_n(\omega y)Y_n'(\omega x)] \tag{42-c}$$

$$F_n^{(4)}(\omega, x, y) = \begin{cases} [J_0'(\omega x)Y_0'(\omega y) - J_0'(\omega y)Y_0'(\omega x)] & n = 0 \\ \omega^2 xy[J_n'(\omega x)Y_n'(\omega y) - J_n'(\omega y)Y_n'(\omega x)] & n \neq 0 \end{cases} \tag{42-d}$$

where a prime denotes differentiation with respect to the argument of the function. The properties of these functions used in this analysis are given in Appendix II.

The functions appearing in the modal solutions (40) and (41) are given below explicitly in terms of the cross products of the Bessel functions for the three cases n = 0, n = 1 and n ≥ 2.

For n = 0 the eigenvalues (ω_{0j}; j = 0, 1, 2, . . .) are the positive real roots of the equation

$$D_0(\omega) = 0. \tag{43}$$

The functions used in the modal solutions are

$$D_0(\omega) = \frac{\pi}{2}\left[\omega^2 F_0^{(1)}(\omega,x,y) + \frac{2\gamma^2}{y^2} F_0^{(2)}(\omega,x,y) + \frac{2\gamma^2}{x^2} F_0^{(3)}(\omega,x,y) + \frac{4\gamma^4}{xy} F_0^{(4)}(\omega,x,y)\right] \tag{44-a}$$

$$C_0^{\varphi}(r,\omega) = \frac{\pi}{2}\left[\omega^2 F_0^{(1)}(\omega,r,y) + \frac{2\gamma^2}{y^2} F_0^{(2)}(\omega,r,y)\right] \tag{44-b}$$

$$\frac{\partial}{\partial r} C_0^{\varphi}(r,\omega) = \frac{\omega^2\pi}{2}\left[\frac{1}{r} F_0^{(3)}(\omega,r,y) + \frac{2\gamma^2}{y} F_0^{(4)}(\omega,r,y)\right] \tag{44-c}$$

$$G_0(\omega) = -\frac{\pi}{2}\left[(1-2\gamma^2)F_0^{(1)}(\omega,x,y) + \frac{1}{2}\left(1 - \frac{4\gamma^4}{\omega^2 y^2}\right)F_0^{(2)}(\omega,x,y) + \frac{1}{2}\left(1 - \frac{4\gamma^4}{\omega^2 x^2}\right)F_0^{(3)}(\omega,x,y) + \gamma^2\left(\frac{x}{y} + \frac{y}{x} - \frac{4\gamma^2}{\omega^2 xy}\right)F_0^{(4)}(\omega,x,y)\right] \tag{44-d}$$

$$W_0^{(s)}(r,t) = -\frac{\gamma^2 r + (1-\gamma^2)\frac{y^2}{r}}{2\gamma^2(1-\gamma^2)\left(1 - \frac{y^2}{x^2}\right)} \tag{45-a}$$

$$S_0^{r(s)}(r,t) = \frac{1 - \frac{y^2}{r^2}}{1 - \frac{y^2}{x^2}} \tag{45-b}$$

$$S_o^{\theta(s)}(r,t) = \frac{1 + \dfrac{y^2}{r^2}}{1 - \dfrac{y^2}{x^2}} \tag{45-c}$$

For n = 1 the eigenvalues $(\omega_{1j};\ j = 0, 1, 2, \ldots)$ are the positive real roots of the equation

$$D_1(\omega) = 0. \tag{46}$$

The corresponding functions used in the modal solutions are

$$D_1(\omega) = -\frac{1}{\omega^2} A_1^{\Delta}(\omega) \tag{47-a}$$

$$C_1^{\varphi}(r,\omega) = -\frac{1}{\omega^2} A_1^{\varphi}(r,\omega) \quad , \quad \frac{\partial C_1^{\varphi}}{\partial r} = -\frac{1}{\omega^2} A_1^{\varphi\,\prime}(r,\omega) \tag{47-b}$$

$$C_1^{\psi}(r,\omega) = -\frac{1}{\omega^2} A_1^{\psi}(r,\omega) \quad , \quad \frac{\partial C_1^{\psi}}{\partial r} = -\frac{1}{\omega^2} A_1^{\psi\,\prime}(r,\omega) \tag{47-c}$$

$$G_1(\omega) = \frac{1}{2\omega^3}\left[\dot{A}_1^{\Delta}(\omega) - \frac{2}{\omega} A_1^{\Delta}(\omega)\right] \tag{47-d}$$

where a prime denotes partial differentiation with respect to r and a dot denotes differentiation with respect to ω. A_1^{φ}, A_1^{ψ}, A_1^{Δ} etc. will be defined shortly.

$$\begin{aligned} W_1^{(s)}(r,t) = \frac{x}{4\alpha}\Bigg\{ t^2 + \frac{y^2}{4}\Bigg[& 3\frac{r^2}{y^2} - 2\left(\frac{1+\gamma^2}{\gamma^2}\right)\ln\frac{r}{y} \\ & + \frac{x^2}{x^2+y^2}\left(\frac{y^2}{r^2} - \left(\frac{1-3\gamma^2}{1-\gamma^2}\right)\frac{r^2}{y^2}\right) \\ & + \frac{(1+\gamma^2)x^2}{\alpha\gamma^2}\ln\frac{x}{y} - 3\Bigg]\Bigg\} \end{aligned} \tag{48-a}$$

$$\begin{aligned} V_1^{(s)}(r,t) = \frac{x}{4\alpha}\Bigg\{ t^2 + \frac{y^2}{4}\Bigg[& \frac{r^2}{y^2} - 2\left(\frac{1+\gamma^2}{\gamma^2}\right)\ln\frac{r}{y} \\ & - \frac{x^2}{x^2+y^2}\left(\frac{y^2}{r^2} + \left(\frac{3-\gamma^2}{1-\gamma^2}\right)\frac{r^2}{y^2}\right) \\ & + \frac{(1+\gamma^2)x^2}{\alpha\gamma^2}\ln\frac{x}{y} - 2\left(\frac{1-\gamma^2}{\gamma^2}\right) - 3\Bigg]\Bigg\} \end{aligned} \tag{48-b}$$

$$S_1^{r(s)}(r,t) = \frac{xy}{4\alpha}\left[(2-\gamma^2)\left(\frac{r}{y}-\frac{y}{r}\right) - \frac{\gamma^2 x^2}{x^2+y^2}\left(\frac{y^3}{r^3}-\frac{r}{y}\right)\right] \tag{48-c}$$

$$S_1^{\theta(s)}(r,t) = \frac{xy}{4\alpha}\left[(2-3\gamma^2)\frac{r}{y} + \gamma^2\frac{y}{r} + \frac{\gamma^2 x^2}{x^2+y^2}\left(\frac{y^3}{r^3}+3\frac{r}{y}\right)\right] \tag{48-d}$$

$$T_1^{(s)}(r,t) = \frac{\gamma^2 xy}{4\alpha}\left[\left(\frac{r}{y}-\frac{y}{r}\right) + \frac{x^2}{x^2+y^2}\left(\frac{y^3}{r^3}-\frac{r}{y}\right)\right] \tag{48-e}$$

For $n = \{2,3,4,\ldots,N\}$ the eigenvalues (ω_{nj} ; $j = 0,1,2,\ldots$) are the positive, real roots of the equation

$$D_n(\omega) = 0. \tag{49}$$

The functions needed for the modal solution are

$$D_n(\omega) = A_n^{\Delta}(\omega) + (n^2-1)B_n^{\Delta}(\omega) \tag{50-a}$$

$$C_n^{\varphi}(r,\omega) = A_n^{\varphi}(r,\omega) + (n^2-1)B_n^{\varphi}(r,\omega) \tag{50-b}$$

$$C_n^{\psi}(r,\omega) = A_n^{\psi}(r,\omega) + (n^2-1)B_n^{\psi}(r,\omega) \tag{50-c}$$

$$\frac{\partial C_n^{\varphi}}{\partial r} = A_n^{\varphi\prime}(r,\omega) + (n^2-1)B_n^{\varphi\prime}(r,\omega) \tag{50-d}$$

$$\frac{\partial C_n^{\psi}}{\partial r} = A_n^{\psi\prime}(r,\omega) + (n^2-1)B_n^{\psi\prime}(r,\omega) \tag{50-e}$$

$$G_n(\omega) = -\frac{1}{2\omega}\left[\dot{A}_n^{\Delta}(\omega) + (n^2-1)\dot{B}_n^{\Delta}(\omega)\right] \tag{50-f}$$

where a prime is used to denote partial differentiation with respect to r and a dot denotes differentiation with respect to ω. A_n^{Δ}, A_n^{φ}, etc. will be defined shortly.

Let

$$a_n = n\left(\frac{x^2}{y^2}-1\right) + \left(\frac{x}{y}\right)^{2n} - 1 \tag{51-a}$$

$$b_n = n\left(1-\frac{y^2}{x^2}\right) + 1 - \left(\frac{y}{x}\right)^{2n} \tag{51-b}$$

$$C_n = n\left(1 - \frac{y^2}{x^2}\right) + \left(\frac{x}{y}\right)^{2n} - 1 \qquad (51\text{-c})$$

$$d_n = n\left(\frac{x^2}{y^2} - 1\right) + 1 - \left(\frac{y}{x}\right)^{2n} \qquad (51\text{-d})$$

$$e_n = \left[n^2\left(\frac{x}{y} - \frac{y}{x}\right)^2 - \left(\frac{x^n}{y^n} - \frac{y^n}{x^n}\right)^2\right]^{-1} \qquad (51\text{-e})$$

then

$$W_n^{(s)}(r,t) = \frac{xe_n}{4\gamma^2}\left\{a_n\left[\frac{n(1-\gamma^2) - 2\gamma^2}{(n+1)(1-\gamma^2)}\right]\left(\frac{r}{x}\right)^{n+1}\right.$$

$$+ b_n\left(\frac{n}{n+1}\right)\left(\frac{x}{r}\right)^{n+1} - C_n\left(\frac{n}{n-1}\right)\left(\frac{r}{x}\right)^{n-1}$$

$$\left. - d_n\left[\frac{n(1-\gamma^2) + 2\gamma^2}{(n-1)(1-\gamma^2)}\right]\left(\frac{x}{r}\right)^{n-1}\right\} \qquad (52\text{-a})$$

$$V_n^{(s)}(r,t) = \frac{xe_n}{4\gamma^2}\left\{a_n\left[\frac{n(1-\gamma^2) + 2}{(n+1)(1-\gamma^2)}\right]\left(\frac{r}{x}\right)^{n+1}\right.$$

$$- b_n\left(\frac{n}{n+1}\right)\left(\frac{x}{r}\right)^{n+1} - C_n\left(\frac{n}{n-1}\right)\left(\frac{r}{x}\right)^{n-1}$$

$$\left. + d_n\left[\frac{n(1-\gamma^2) - 2}{(n-1)(1-\gamma^2)}\right]\left(\frac{x}{r}\right)^{n-1}\right\} \qquad (52\text{-b})$$

$$S_n^{r(s)}(r,t) = \frac{e_n}{2}\left\{a_n(n-2)\left(\frac{r}{x}\right)^n - nb_n\left(\frac{x}{r}\right)^{n+2}\right.$$

$$\left. - nC_n\left(\frac{r}{x}\right)^{n-2} + d_n(n+2)\left(\frac{x}{r}\right)^n\right\} \qquad (52\text{-c})$$

$$S_n^{\theta(s)}(r,t) = \frac{e_n}{2}\left\{- a_n(n+2)\left(\frac{r}{x}\right)^n + nb_n\left(\frac{x}{r}\right)^{n+2}\right.$$

$$\left. + nC_n\left(\frac{r}{x}\right)^{n-2} - d_n(n-2)\left(\frac{x}{r}\right)^n\right\} \qquad (52\text{-d})$$

$$T_n^{(s)}(r,t) = \frac{ne_n}{2}\left\{a_n\left(\frac{r}{x}\right)^n + b_n\left(\frac{x}{r}\right)^{n+2} - C_n\left(\frac{r}{x}\right)^{n-2} - d_n\left(\frac{x}{r}\right)^n\right\} \qquad (52\text{-e})$$

The functions A_n^{Δ}, A_n^{φ}, etc. are given below for $n = \{1, 2, 3, \ldots, N\}$.

$$A_n^{\Delta}(\omega) = \frac{8\gamma^4 n^2}{x^2 y^2} + \frac{\pi^2}{4}\left(\omega^2 - \frac{4\gamma^2 n^2}{x^2}\right)\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right) F_n^{(1)}(\omega, x, y) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, y\right)$$

$$+ \frac{\pi^2}{4}\, \frac{2\gamma^2}{y^2}\left(\omega^2 - \frac{4\gamma^2 n^2}{x^2}\right)\left[F_n^{(1)}(\omega, x, y) F_n^{(2)}\left(\frac{\omega}{\gamma}, x, y\right) + F_n^{(2)}(\omega, x, y) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, y\right)\right]$$

$$+ \frac{\pi^2}{4}\, \frac{2\gamma^2}{x^2}\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right)\left[F_n^{(1)}(\omega, x, y) F_n^{(3)}\left(\frac{\omega}{\gamma}, x, y\right) + F_n^{(3)}(\omega, x, y) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, y\right)\right]$$

$$+ \frac{\pi^2}{4}\, \frac{4\gamma^4}{x^2 y^2}\left[F_n^{(1)}(\omega, x, y) F_n^{(4)}\left(\frac{\omega}{\gamma}, x, y\right) + F_n^{(4)}(\omega, x, y) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, y\right)\right]$$

$$+ \frac{\pi^2}{4}\, \frac{4\gamma^4}{x^2 y^2}\left[F_n^{(2)}(\omega, x, y) F_n^{(3)}\left(\frac{\omega}{\gamma}, x, y\right) + F_n^{(3)}(\omega, x, y) F_n^{(2)}\left(\frac{\omega}{\gamma}, x, y\right)\right] \tag{53-a}$$

$$A_n^{\varphi}(r, \omega) = \frac{\pi}{2}\, \frac{4\gamma^4 n^2}{x^2 y^2}\left[F_n^{(1)}(\omega, x, r) - F_n^{(3)}(\omega, x, r)\right]$$

$$+ \frac{\pi^2}{4}\left(\omega^2 - \frac{2\gamma^2 n^2}{x^2}\right)\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right) F_n^{(1)}(\omega, r, y) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, y\right)$$

$$+ \frac{\pi^2}{4}\, \frac{2\gamma^2}{y^2}\left(\omega^2 - \frac{2\gamma^2 n^2}{x^2}\right)\left[F_n^{(1)}(\omega, r, y) F_n^{(2)}\left(\frac{\omega}{\gamma}, x, y\right) + F_n^{(2)}(\omega, r, y) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, y\right)\right]$$

$$+ \frac{\pi^2}{4}\, \frac{2\gamma^2}{x^2}\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right) F_n^{(1)}(\omega, r, y) F_n^{(3)}\left(\frac{\omega}{\gamma}, x, y\right)$$

$$+ \frac{\pi^2}{4}\, \frac{4\gamma^4}{x^2 y^2}\left[F_n^{(1)}(\omega, r, y) F_n^{(4)}\left(\frac{\omega}{\gamma}, x, y\right) + F_n^{(2)}(\omega, r, y) F_n^{(3)}\left(\frac{\omega}{\gamma}, x, y\right)\right] \tag{53-b}$$

$$A_n^{\psi}(r, \omega) = \frac{\pi}{2}\, \frac{2\gamma^2 n}{y^2}\left[-\left(\omega^2 - \frac{2\gamma^2 n^2}{x^2}\right) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, r\right) - \frac{2\gamma^2}{x^2} F_n^{(3)}\left(\frac{\omega}{\gamma}, x, r\right)\right]$$

$$- \frac{\pi^2}{4}\, \frac{2\gamma^2 n}{x^2}\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right) F_n^{(1)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(1)}(\omega, x, y)$$

$$- \frac{\pi^2}{4}\, \frac{4\gamma^4 n}{x^2 y^2}\left[F_n^{(1)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(2)}(\omega, x, y) + F_n^{(2)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(1)}(\omega, x, y)\right]$$

$$+\frac{\pi^2}{4}\frac{2\gamma^2 n}{x^2}\left(\omega^2-\frac{4\gamma^2 n^2}{y^2}\right)F_n^{(1)}\left(\frac{\omega}{\gamma},r,y\right)F_n^{(3)}(\omega,x,y)$$

$$+\frac{\pi^2}{4}\frac{4\gamma^4 n}{x^2y^2}\left[F_n^{(1)}\left(\frac{\omega}{\gamma},r,y\right)F_n^{(4)}(\omega,x,y)+F_n^{(2)}\left(\frac{\omega}{\gamma},r,y\right)F_n^{(3)}(\omega,x,y)\right]$$

(53-c)

$$B_n^{\Delta}(\omega)=\frac{\pi^2\gamma^4}{\omega^4x^2y^2}\left\{\frac{4}{\pi^2}\left[\frac{8\gamma^4n^2(n^2-1)}{x^2y^2}-4\gamma^2n^2\omega^2\left(\frac{1}{y^2}+\frac{1}{x^2}\right)\right]\right.$$

$$+n^2\left[\frac{4\gamma^4n^2(n^2-1)}{x^2y^2}-4\gamma^2n^2\omega^2\left(\frac{1}{y^2}+\frac{1}{x^2}\right)+\omega^4\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)\right]\cdot$$

$$\cdot F_n^{(1)}(\omega,x,y)F_n^{(1)}\left(\frac{\omega}{\gamma},x,y\right)$$

$$-\frac{x^2}{y^2}\left[\frac{4\gamma^4n^2(n^2-1)}{x^4}-\frac{4\gamma^2n^2\omega^2}{x^2}+\omega^4\right]F_n^{(2)}(\omega,x,y)F_n^{(2)}\left(\frac{\omega}{\gamma},x,y\right)$$

$$-\frac{y^2}{x^2}\left[\frac{4\gamma^4n^2(n^2-1)}{y^4}-\frac{4\gamma^2n^2\omega^2}{y^2}+\omega^4\right]F_n^{(3)}(\omega,x,y)F_n^{(3)}\left(\frac{\omega}{\gamma},x,y\right)$$

$$+\frac{4\gamma^4(n^2-1)}{x^2y^2}F_n^{(4)}(\omega,x,y)F_n^{(4)}\left(\frac{\omega}{\gamma},x,y\right)$$

$$+\frac{2\gamma^2n^2\omega^2}{x^2}\left[F_n^{(1)}(\omega,x,y)F_n^{(2)}\left(\frac{\omega}{\gamma},x,y\right)+F_n^{(2)}(\omega,x,y)F_n^{(1)}\left(\frac{\omega}{\gamma},x,y\right)\right]$$

$$+\frac{2\gamma^2n^2\omega^2}{y^2}\left[F_n^{(1)}(\omega,x,y)F_n^{(3)}\left(\frac{\omega}{\gamma},x,y\right)+F_n^{(3)}(\omega,x,y)F_n^{(1)}\left(\frac{\omega}{\gamma},x,y\right)\right]$$

$$-\frac{2\gamma^2\omega^2}{x^2}\left[F_n^{(3)}(\omega,x,y)F_n^{(4)}\left(\frac{\omega}{\gamma},x,y\right)+F_n^{(4)}(\omega,x,y)F_n^{(3)}\left(\frac{\omega}{\gamma},x,y\right)\right]$$

$$\left.-\frac{2\gamma^2\omega^2}{y^2}\left[F_n^{(2)}(\omega,x,y)F_n^{(4)}\left(\frac{\omega}{\gamma},x,y\right)+F_n^{(4)}(\omega,x,y)F_n^{(2)}\left(\frac{\omega}{\gamma},x,y\right)\right]\right\}$$

(53-d)

$$B_n^{\omega}(r,\omega)=-\frac{\pi^2\gamma^4}{\omega^2x^2y^4}\left\{-\frac{4\gamma^2n^2}{\pi}\left[F_n^{(3)}(\omega,x,r)-F_n^{(1)}(\omega,x,r)\right]\right.$$

$$-n^2(\omega^2x^2-2\gamma^2n^2)F_n^{(1)}(\omega,r,y)F_n^{(1)}\left(\frac{\omega}{\gamma},x,y\right)$$

$$- 2\gamma^2 n^2 F_n^{(1)}(\omega, r, y) F_n^{(3)}\left(\frac{\omega}{\gamma}, x, y\right)$$

$$+ (\omega^2 x^2 - 2\gamma^2 n^2) F_n^{(2)}(\omega, r, y) F_n^{(2)}\left(\frac{\omega}{\gamma}, x, y\right)$$

$$+ 2\gamma^2 F_n^{(2)}(\omega, r, y) F_n^{(4)}\left(\frac{\omega}{\gamma}, x, y\right)\Bigg\} \qquad (53\text{-e})$$

$$B_n^{\psi}(r, \omega) = - \frac{\pi^2 \gamma^4 n}{\omega^2 x^2 y^4} \Bigg\{ - \frac{2}{\pi}\left[2\gamma^2 F_n^{(3)}\left(\frac{\omega}{\gamma}, x, r\right) + \left(\omega^2 x^2 - 2\gamma^2 n^2\right) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, r\right)\right]$$

$$+ 2\gamma^2 n^2 F_n^{(1)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(1)}(\omega, x, y)$$

$$- 2\gamma^2 n^2 F_n^{(1)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(3)}(\omega, x, y)$$

$$- 2\gamma^2 F_n^{(2)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(2)}(\omega, x, y)$$

$$+ 2\gamma^2 F_n^{(2)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(4)}(\omega, x, y)\Bigg\} \qquad (53\text{-f})$$

$$A_n^{\varphi\prime}(r, \omega) = \frac{\pi^2}{4} \frac{1}{r} \Bigg\{ \frac{2}{\pi} \frac{4\gamma^4 n^2}{x^2 y^2} \left[F_n^{(2)}(\omega, x, r) - F_n^{(4)}(\omega, x, r)\right]$$

$$+ \left(\omega^2 - \frac{2\gamma^2 n^2}{x^2}\right)\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right) F_n^{(3)}(\omega, r, y) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, y\right)$$

$$+ \frac{2\gamma^2}{y^2}\left(\omega^2 - \frac{2\gamma^2 n^2}{x^2}\right)\left[F_n^{(3)}(\omega, r, y) F_n^{(2)}\left(\frac{\omega}{\gamma}, x, y\right) + F_n^{(4)}(\omega, r, y) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, y\right)\right]$$

$$+ \frac{2\gamma^2}{x^2}\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right) F_n^{(3)}(\omega, r, y) F_n^{(3)}\left(\frac{\omega}{\gamma}, x, y\right)$$

$$+ \frac{4\gamma^4}{x^2 y^2}\left[F_n^{(3)}(\omega, r, y) F_n^{(4)}\left(\frac{\omega}{\gamma}, x, y\right) + F_n^{(4)}(\omega, r, y) F_n^{(3)}\left(\frac{\omega}{\gamma}, x, y\right)\right]\Bigg\} \qquad (54\text{-a})$$

$$A_n^{\psi\prime}(r, \omega) = \frac{\pi^2}{4} \frac{1}{r} \Bigg\{ \frac{2}{\pi} \frac{2\gamma^2 n}{y^2}\left[-\left(\omega^2 - \frac{2\gamma^2 n^2}{x^2}\right) F_n^{(2)}\left(\frac{\omega}{\gamma}, x, r\right) - \frac{2\gamma^2}{x^2} F_n^{(4)}\left(\frac{\omega}{\gamma}, x, r\right)\right]$$

$$- \frac{2\gamma^2 n}{x^2}\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right) F_n^{(3)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(1)}(\omega, x, y)$$

$$- \frac{4\gamma^4 n}{x^2 y^2}\left[F_n^{(3)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(2)}(\omega, x, y) + F_n^{(4)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(1)}(\omega, x, y)\right]$$

$$+ \frac{2\gamma^2 n}{x^2}\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right) F_n^{(3)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(3)}(\omega, x, y)$$

$$+ \frac{4\gamma^4 n}{x^2 y^2}\left[F_n^{(3)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(4)}(\omega, x, y) + F_n^{(4)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(3)}(\omega, x, y)\right]\Bigg\}$$

(54-b)

$$B_n^{\varphi\prime}(r, \omega) = - \frac{\pi^2 \gamma^4}{\omega^2 x^2 y^4}\frac{1}{r}\Bigg\{- \frac{4\gamma^2 n^2}{\pi}\left[F_n^{(4)}(\omega, x, r) - F_n^{(2)}(\omega, x, r)\right]$$

$$- n^2(\omega^2 x^2 - 2\gamma^2 n^2) F_n^{(3)}(\omega, r, y) F_n^{(1)}\left(\frac{\omega}{\gamma}, x, y\right)$$

$$- 2\gamma^2 n^2 F_n^{(3)}(\omega, r, y) F_n^{(3)}\left(\frac{\omega}{\gamma}, x, y\right)$$

$$+ (\omega^2 x^2 - 2\gamma^2 n^2) F_n^{(4)}(\omega, r, y) F_n^{(2)}\left(\frac{\omega}{\gamma}, x, y\right)$$

$$+ 2\gamma^2 F_n^{(4)}(\omega, r, y) F_n^{(4)}\left(\frac{\omega}{\gamma}, x, y\right)\Bigg\}$$

(54-c)

$$B_n^{\psi\prime}(r, \omega) = - \frac{\pi^2 \gamma^4 n}{\omega^2 x^2 y^4}\frac{1}{r}\Bigg\{- \frac{2}{\pi}\left[2\gamma^2 F_n^{(4)}\left(\frac{\omega}{\gamma}, x, r\right) + (\omega^2 x^2 - 2\gamma^2 n^2) F_n^{(2)}\left(\frac{\omega}{\gamma}, x, r\right)\right]$$

$$+ 2\gamma^2 n^2 F_n^{(3)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(1)}(\omega, x, y)$$

$$- 2\gamma^2 n^2 F_n^{(3)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(3)}(\omega, x, y)$$

$$- 2\gamma^2 F_r^{(4)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(2)}(\omega, x, y)$$

$$+ 2\gamma^2 F_n^{(4)}\left(\frac{\omega}{\gamma}, r, y\right) F_n^{(4)}(\omega, x, y)\Bigg\}$$

(54-d)

$$\dot{I}_n^{\Delta}(\omega) = \frac{\pi^2}{4}\Bigg\{2\omega\left[2\omega^2 - 4\gamma^2 n^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right)\right] F_n^{(1,1)}(\omega)$$

$$+ \left(\omega^2 - \frac{4\gamma^2 n^2}{x^2}\right)\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right) \dot{F}_n^{(1,1)}(\omega)$$

$$+ 2\omega \frac{4\gamma^2}{y^2} F_n^{(1,2)}(\omega) + \frac{4\gamma^2}{y^2}\left(\omega^2 - \frac{4\gamma^2 n^2}{x^2}\right) \dot{F}_n^{(1,2)}(\omega)$$

$$+ 2\omega \frac{4\gamma^2}{x^2} F_n^{(1,3)}(\omega) + \frac{4\gamma^2}{x^2}\left(\omega^2 - \frac{4\gamma^2 n^2}{y^2}\right) \dot{F}_n^{(1,3)}(\omega)$$

$$+ \frac{8\gamma^4}{x^2 y^2} \left[\dot{F}_n^{(1,4)}(\omega) + \dot{F}_n^{(2,3)}(\omega) \right] \Bigg\} \tag{55-a}$$

$$\dot{B}_n^{\Delta}(\omega) = - \frac{4}{\omega} B_n^{\Delta}(\omega) + \frac{\pi^3 \gamma^4}{\omega^4 x^2 y^2} \Bigg\{ - 2\omega \frac{16\gamma^2 n^2}{\pi^2} \left(\frac{1}{y^2} + \frac{1}{x^2} \right)$$

$$n^2 \left[4\omega^3 \left(\frac{x^2}{y^2} + \frac{y^2}{x^2} \right) - 8\omega\gamma^2 n^2 \left(\frac{1}{y^2} + \frac{1}{x^2} \right) \right] F_n^{(1,1)}(\omega)$$

$$+ n^2 \left[\frac{4\gamma^4 n^2 (n^2 - 1)}{x^2 y^2} - 4\gamma^2 n^2 \omega^2 \left(\frac{1}{y^2} + \frac{1}{x^2} \right) + \omega^4 \left(\frac{x^2}{y^2} + \frac{y^2}{x^2} \right) \right] \dot{F}_n^{(1,1)}(\omega)$$

$$- \left[\frac{4\gamma^4 n^2 (n^2 - 1)}{x^2 y^2} - 4\gamma^2 n^2 \omega^2 \frac{1}{y^2} + \omega^4 \frac{x^2}{y^2} \right] \dot{F}_n^{(2,2)}(\omega)$$

$$- \left[4\omega^3 \frac{x^2}{y^2} - 8\omega\gamma^2 n^2 \frac{1}{y^2} \right] F_n^{(2,2)}(\omega)$$

$$- \left[\frac{4\gamma^4 n^2 (n^2 - 1)}{x^2 y^2} - 4\gamma^2 n^2 \omega^2 \frac{1}{x^2} + \omega^4 \frac{y^2}{x^2} \right] \dot{F}_n^{(3,3)}(\omega)$$

$$- \left[4\omega^3 \frac{y^2}{x^2} - 8\omega\gamma^2 n^2 \frac{1}{x^2} \right] F_n^{(3,3)}(\omega)$$

$$+ \frac{4\gamma^4 (n^2 - 1)}{x^2 y^2} \dot{F}_n^{(4,4)}(\omega)$$

$$+ \frac{8\omega\gamma^2 n^2}{x^2} F_n^{(1,2)}(\omega) + \frac{4\gamma^2 n^2 \omega^2}{x^2} \dot{F}_n^{(1,2)}(\omega)$$

$$+ \frac{8\omega\gamma^2 n^2}{y^2} F_n^{(1,3)}(\omega) + \frac{4\gamma^2 n^2 \omega^2}{y^2} \dot{F}_n^{(1,3)}(\omega)$$

$$- \frac{8\omega\gamma^2}{x^2} F_n^{(3,4)}(\omega) - \frac{4\gamma^2 \omega^2}{x^2} \dot{F}_n^{(3,4)}(\omega)$$

$$- \frac{8\omega\gamma^2}{y^2} F_n^{(2,4)}(\omega) - \frac{4\gamma^2 \omega^2}{y^2} \dot{F}_n^{(2,4)}(\omega) \Bigg\} \tag{55-b}$$

where

$$\dot{F}_n^{(1,1)}(\omega) = \frac{2}{\omega} \left[F_n^{(1,2)}(\omega) + F_n^{(1,3)}(\omega) \right] \tag{56-a}$$

$$\dot{F}_n^{(1,2)}(\omega) = \frac{1}{\omega} \left[F_n^{(1,4)}(\omega) + F_n^{(2,3)}(\omega) + F_n^{(2,2)}(\omega) - \left(\omega^2 y^2 \frac{1 + \gamma^2}{2\gamma^2} - n^2 \right) F_n^{(1,1)}(\omega) \right] \tag{56-b}$$

$$\dot{F}_n^{(1,3)}(\omega) = \frac{1}{\omega}\Big[F_n^{(1,4)}(\omega) + F_n^{(2,3)}(\omega) + F_n^{(3,3)}(\omega) - \Big(\omega^2 x^2 \frac{1+\gamma^2}{2\gamma^2} - n^2\Big) F_n^{(1,1)}(\omega)\Big]$$

(56-c)

$$\dot{F}_n^{(1,4)}(\omega) = \frac{1}{\omega}\Bigg\{F_n^{(2,4)}(\omega) + F_n^{(3,4)}(\omega) + n^2 F_n^{(1,2)}(\omega) + n^2 F_n^{(1,3)}(\omega)$$
$$- \frac{\omega^2 x^2}{2}\Big[\frac{1}{\gamma^2} F_n^{(1)}(\omega,x,y) F_n^{(2)}\Big(\frac{\omega}{\gamma},x,y\Big) + F_n^{(1)}\Big(\frac{\omega}{\gamma},x,y\Big) F_n^{(2)}(\omega,x,y)\Big]$$
$$- \frac{\omega^2 y^2}{2}\Big[\frac{1}{\gamma^2} F_n^{(1)}(\omega,x,y) F_n^{(3)}\Big(\frac{\omega}{\gamma},x,y\Big) + F_n^{(1)}\Big(\frac{\omega}{\gamma},x,y\Big) F_n^{(3)}(\omega,x,y)\Big]\Bigg\}$$

(56-d)

$$\dot{F}_n^{(2,2)}(\omega) = \frac{2}{\omega}\Bigg\{F_n^{(2,4)}(\omega) + n^2 F_n^{(1,2)}(\omega)$$
$$- \frac{\omega^2 y^2}{2}\Big[\frac{1}{\gamma^2} F_n^{(2)}(\omega,x,y) F_n^{(1)}\Big(\frac{\omega}{\gamma},x,y\Big) + F_n^{(1)}(\omega,x,y) F_n^{(2)}\Big(\frac{\omega}{\gamma},x,y\Big)\Big]\Bigg\}$$

(56-e)

$$\dot{F}_n^{(2,3)}(\omega) = \frac{1}{\omega}\Bigg\{F_n^{(2,4)}(\omega) + F_n^{(3,4)}(\omega) + n^2 F_n^{(1,2)}(\omega) + n^2 F_n^{(1,3)}(\omega)$$
$$- \frac{\omega^2 x^2}{2}\Big[\frac{1}{\gamma^2} F_n^{(2)}(\omega,x,y) F_n^{(1)}\Big(\frac{\omega}{\gamma},x,y\Big) + F_n^{(1)}(\omega,x,y) F_n^{(2)}\Big(\frac{\omega}{\gamma},x,y\Big)\Big]$$
$$- \frac{\omega^2 y^2}{2}\Big[\frac{1}{\gamma^2} F_n^{(3)}(\omega,x,y) F_n^{(1)}\Big(\frac{\omega}{\gamma},x,y\Big) + F_n^{(1)}(\omega,x,y) F_n^{(3)}\Big(\frac{\omega}{\gamma},x,y\Big)\Big]\Bigg\}$$

(56-f)

$$\dot{F}_n^{(2,4)}(\omega) = \frac{1}{\omega}\Bigg\{-\Big(\omega^2 x^2 \frac{1+\gamma^2}{2\gamma^2} - n^2\Big) F_n^{(2,2)}(\omega) + F_n^{(4,4)}(\omega) + n^2 F_n^{(2,3)}(\omega)$$
$$+ n^2 F_n^{(1,4)}(\omega) - \frac{\omega^2 y^2}{2}\Big[\frac{1}{\gamma^2} F_n^{(2)}(\omega,x,y) F_n^{(3)}\Big(\frac{\omega}{\gamma},x,y\Big)$$
$$+ F_n^{(3)}(\omega,x,y) F_n^{(2)}\Big(\frac{\omega}{\gamma},x,y\Big) + \frac{1}{\gamma^2} F_n^{(4)}(\omega,x,y) F_n^{(1)}\Big(\frac{\omega}{\gamma},x,y\Big)$$
$$+ F_n^{(1)}(\omega,x,y) F_n^{(4)}\Big(\frac{\omega}{\gamma},x,y\Big)\Big]\Bigg\}$$

(56-g)

$$\dot{F}_n^{(3,3)}(\omega) = \frac{2}{\omega}\Bigg\{F_n^{(3,4)}(\omega) + n^2 F_n^{(1,3)}(\omega)$$
$$- \frac{\omega^2 x^2}{2}\Big[\frac{1}{\gamma^2} F_n^{(3)}(\omega,x,y) F_n^{(1)}\Big(\frac{\omega}{\gamma},x,y\Big) + F_n^{(1)}(\omega,x,y) F_n^{(3)}\Big(\frac{\omega}{\gamma},x,y\Big)\Big]\Bigg\}$$

(56-h)

$$\dot{F}_n^{(3,4)}(\omega) = \frac{1}{\omega}\Bigg\{-\left(\omega^2 y^2 \frac{1+\gamma^2}{2\gamma^2} - n^2\right) F_n^{(3,3)}(\omega) + F_n^{(4,4)}(\omega) + n^2 F_n^{(2,3)}(\omega)$$

$$+ n^2 F_n^{(1,4)}(\omega) - \frac{\omega^2 x^2}{2}\left[\frac{1}{\gamma^2} F_n^{(3)}(\omega,x,y) F_n^{(2)}\left(\frac{\omega}{\gamma},x,y\right)\right.$$

$$+ F_n^{(2)}(\omega,x,y) F_n^{(3)}\left(\frac{\omega}{\gamma},x,y\right) + \frac{1}{\gamma^2} F_n^{(4)}(\omega,x,y) F_n^{(1)}\left(\frac{\omega}{\gamma},x,y\right)$$

$$\left. + F_n^{(1)}(\omega,x,y) F_n^{(4)}\left(\frac{\omega}{\gamma},x,y\right)\right]\Bigg\} \tag{56-i}$$

$$\dot{F}_n^{(4,4)}(\omega) = \frac{2}{\omega}\Bigg\{n^2 F_n^{(2,4)}(\omega) + n^2 F_n^{(3,4)}(\omega)$$

$$- \frac{\omega^2 x^2}{2}\left[\frac{1}{\gamma^2} F_n^{(4)}(\omega,x,y) F_n^{(2)}\left(\frac{\omega}{\gamma},x,y\right) + F_n^{(4)}\left(\frac{\omega}{\gamma},x,y\right) F_n^{(2)}(\omega,x,y)\right]$$

$$- \frac{\omega^2 y^2}{2}\left[\frac{1}{\gamma^2} F_n^{(4)}(\omega,x,y) F_n^{(3)}\left(\frac{\omega}{\gamma},x,y\right) + F_n^{(4)}\left(\frac{\omega}{\gamma},x,y\right) F_n^{(3)}(\omega,x,y)\right]\Bigg\} \tag{56-j}$$

where

$$F_n^{(i,j)}(\omega) = \frac{1}{2}\left[F_n^{(i)}(\omega,x,y) F_n^{(j)}\left(\frac{\omega}{\gamma}, x,y\right)\right.$$

$$\left. + F_n^{(i)}\left(\frac{\omega}{\gamma},x,y\right) F_n^{(j)}(\omega,x,y)\right]$$

The completes the analytical solution of the equations of the three dimensional theory of elasticity for the stated problem. In the next section some observations of the form of these solutions will be offered.

C. Discussion of the Elasticity Solution

The most conspicuous attribute of the solutions given by (17), (18) and (40) is their double series form. The sums shown in (17) and (18) are over all angular harmonics (n) contained in the load representation. Since the load, by earlier agreement, contains a finite number of harmonic components (N), the sums (17) and (18) are finite. However, from (40) it is observed that each harmonic component or mode of the response is itself an infinite series. To establish the convergence of these series a brief study of the asymptotic behavior of the characteristic values ω_{nj} and the characteristic functions W_{nj}, V_{nj} etc. for large values of the summation index j will now be presented. The following results may be deduced from the asymptotic behavior of the cross products of the Bessel functions for large values of ω given in Appendix II.

$$D_o(\omega) \sim - \frac{\omega}{\sqrt{XY}} \sin \alpha\omega \tag{57-a}$$

$$D_1(\omega) \sim \frac{-\gamma}{XY} \sin \alpha\omega \sin \frac{\alpha\omega}{\gamma} \tag{57-b}$$

$$D_n(\omega) \sim \frac{\gamma\omega^2}{XY} \sin \alpha\omega \sin \frac{\alpha\omega}{\gamma} \ ; \ n \geq 2 \tag{57-c}$$

Since the characteristic values ω_{nj} are the roots of the equation $D_n(\omega) = 0$ we conclude from (57) that

$$\omega_{oj} \sim \frac{K\pi}{\alpha} \ , \ K = 1, 2, 3, \ldots \tag{58-a}$$

$$\omega_{nj} \sim \frac{K\pi}{\alpha} \ , \ \gamma \frac{L\pi}{\alpha} \ ; \ (K, L) = 1, 2, 3, \ldots \tag{58-b}$$

For any given value of n, the integers K and L are linearly related to j for all j above some minimum value J(n). The formulae (58) will be very useful when calculating the characteristic values numerically. Gazis [7] also obtained the relations (57) in his study of hollow cylinders. Using these approximations for

ω_{nj} the series (40) are found to behave asymptotically for large j as follows.

For n = 0, 1, 2, . . .

$$W_{nj}(r)Q_{nj}(t) \sim \frac{2\alpha}{\pi^2}\sqrt{\frac{X}{r}}\;\frac{(-1)^{K+1}}{K^2}\cos\frac{K\pi}{\alpha}(r-Y)\cos\frac{K\pi}{\alpha}t \tag{59-a}$$

$$S^r_{nj}(r)Q_{nj}(t) \sim \frac{2}{\pi}\sqrt{\frac{X}{r}}\;\frac{(-1)^K}{K}\sin\frac{K\pi}{\alpha}(r-Y)\cos\frac{K\pi}{\alpha}t \tag{59-b}$$

$$S^\theta_{nj}(r)Q_{nj}(t) \sim (1-2\gamma^2)\,\frac{2}{\pi}\sqrt{\frac{X}{r}}\;\frac{(-1)^K}{K}\sin\frac{K\pi}{\alpha}(r-Y)\cos\frac{K\pi}{\alpha}t \tag{59-c}$$

For n = 1, 2, 3, . . .

$$V_{nj}(r)Q_{nj}(t) \sim \frac{2\alpha^2 n}{\pi^3}\sqrt{\frac{X}{r}}\;\frac{(-1)^{K+1}}{K^3}\left\{\frac{1}{r}\sin\frac{K\pi}{\alpha}(r-Y)\right.$$

$$\left. + \frac{2\gamma}{Y\sin\frac{K\pi}{\gamma}}\left[\frac{X}{r}\cos\frac{K\pi}{\alpha\gamma}(X-r)+(-1)^K\,\frac{Y}{X}\cos\frac{K\pi}{\alpha\gamma}(r-Y)\right]\right\}\cos\frac{K\pi}{\alpha}t$$

or

$$V_{nj}(r)Q_{nj}(t) \sim \frac{4\alpha^2 n}{\pi^3\gamma Y}\sqrt{\frac{X}{r}}\;\frac{(-1)^{L+1}}{L^3}\;\frac{1}{\sin\gamma L\pi}\left[\frac{X}{r}\cos\frac{L\pi}{\alpha}(X-r)\right.$$

$$\left. + \frac{Y}{X}\cos\gamma L\pi\cos\frac{L\pi}{\alpha}(r-Y)\right]\cos\gamma\frac{L\pi}{\alpha}t \tag{59-d}$$

$$T_{nj}(r)Q_{nj}(t) \sim \frac{4\alpha n\gamma^2}{\pi^2}\sqrt{\frac{X}{r}}\;\frac{(-1)^{K+1}}{K^2}\left\{\frac{1}{r}\cos\frac{K\pi}{\alpha}(r-Y)\right.$$

$$\left. - \frac{1}{\sin\frac{K\pi}{\gamma}}\left[\frac{1}{Y}\sin\frac{K\pi}{\alpha\gamma}(X-r)+\frac{(-1)^K}{X}\sin\frac{K\pi}{\alpha\gamma}(r-Y)\right]\right\}\cos\frac{K\pi}{\alpha}t$$

or

$$T_{nj}(r)Q_{nj}(t) \sim \frac{4\alpha n\gamma}{\pi^2}\sqrt{\frac{X}{r}}\;\frac{(-1)^L}{L^2}\;\frac{1}{\sin\gamma L\pi}\left[\frac{1}{Y}\sin\frac{L\pi}{\alpha}(X-r)\right.$$

$$\left. + \frac{1}{X}\cos\gamma L\pi\sin\frac{L\pi}{\alpha}(r-Y)\right]\cos\gamma\frac{L\pi}{\alpha}t \tag{59-e}$$

In each of the above expressions the first term occurs at the roots $\frac{K\pi}{\alpha}$ and the second at $\gamma \frac{L\pi}{\alpha}$.

We may conclude from the above asymptotic relations that all the series in (40) do in fact converge for $(Y < r < X)$.

In addition to verifying the convergence of the series (40), the relations (59) also provide a useful aid to understanding the nature of the solutions. For example, it is noted that all these relations are either of the form $e^{i\frac{K\pi}{\alpha}(r-Y\pm t)}$ or $e^{i\frac{K\pi}{\alpha}(r-Y+\gamma t)}$. Both represent traveling waves having in the first case a unit phase velocity corresponding to the dimensionless dilatational wave speed and in the second case a phase velocity γ corresponding to the dimensionless shear wave speed. Therefore, we may interpret (59-b, c) as yielding discontinuous dilatational stress waves in $S_n^r(r,t)$ and $S_n^\theta(r,t)$. These discontinuities or steps in the stress are the result of the dilatational wave produced by the suddenly applied load being reflected between the boundaries of the shell. From (59-e) we see that the modal shear stress is composed of both dilatational and shear waves, however they are continuous and in the form of a ramp function rather than a step function. This is to be expected since there are no discontinuities in the shear stress introduced at the boundaries as was the case for the radial stress. From (59-a) we observe that the radial displacement is continuous and dependent primarily on the dilatational wave while the circumferential displacement (59-d) is dependent upon both the dilatational and shear waves.

At this point it will be convenient to consider the practical computation of the series (40) in a specific problem. Since these series converge at least in the manner of a step function, they may be terminated after summing some finite number of terms to obtain an approximation of the desired function. Just as in the load representation, the Lanczos smoothing process may be applied to these finite sums to increase the accuracy of the approximation.

This process, as explained in Appendix I, will result in each term of the sum being multiplied by the factor $\frac{\omega_K}{\omega_j \tau} \sin \frac{\omega_j \tau}{\omega_K}$ where $j \quad K$ is the last term retained in the sum.

It is interesting to note that the form of the solutions (17), (18) and (40) is exactly the same as would be obtained by solving the problem by the Williams or mode acceleration technique [37]. In this context the terms $W_n^{(s)}$, $V_n^{(s)}$, etc. are referred to as the "static" modal solutions, the ω_{nj} are called the natural frequencies of the system and W_{nj}, V_{nj}, etc. the eigenfunctions of the system. In fact, for the suddenly applied load the "static" solution is the solution of the corresponding static problem in which both the equations and boundary conditions are independent of time.

This concludes the analytical investigation of elasticity theory and we will now proceed to examine a specific example for the comparison of the theories.

IV. A NUMERICAL EXAMPLE

In order to obtain a better insight into the nature of the elasticity solution and also a clear comparison with the shell theories a specific shell geometry and load distribution will be studied. An interesting example which also has physical applications is the suddenly loaded cylindrical arch shown in Figure (1-B). A radial load of constant intensity P_o is suddenly applied to the outer surface at $(-\beta < \theta < \beta)$ of a cylindrical arch whose ends at $\theta = \pm \frac{\pi}{2}$ are free to move horizontally on frictionless rollers but restrained from moving vertically. The proper boundary conditions at the ends are $v = \tau_{\theta r} = 0$ at $\theta = \pm \frac{\pi}{2}$. This problem is analogous to the problem of a complete cylindrical shell of identical thickness and material properties subjected to the same loading as the arch plus the symmetric reflection of this load about the $\theta = \pm \frac{\pi}{2}$ planes. In this complete shell, because of the symmetry of the load, $v = \tau_{r\theta} = 0$ at $\theta = \pm \frac{\pi}{2}$. The proper load distribution function for this example is therefore

$$g(\theta) = \begin{Bmatrix} 1 & , & |\theta| \le \beta \\ 0 & , & \beta < |\theta| < \pi - \beta \\ 1 & , & \pi - \beta \le |\theta| \le \pi \end{Bmatrix}$$

The Fourier coefficients of this function are

$$a_o = 4\beta$$

$$a_n = \frac{4}{n} \sin n\beta \qquad ; \quad n = 2, 4, 6, \ldots$$

$$a_n = 0 \qquad ; \quad n = 1, 3, 5, \ldots$$

$$b_n = 0 \qquad ; \quad n = 1, 2, 3, \ldots$$

The thickness ratio used in this example is $\tau = 0.1$. The only material property required in the analysis is Poisson's ratio which was chosen to be $\nu = 0.3$ from which the wave speed ratio $\gamma \cong 0.5345$ may be obtained.

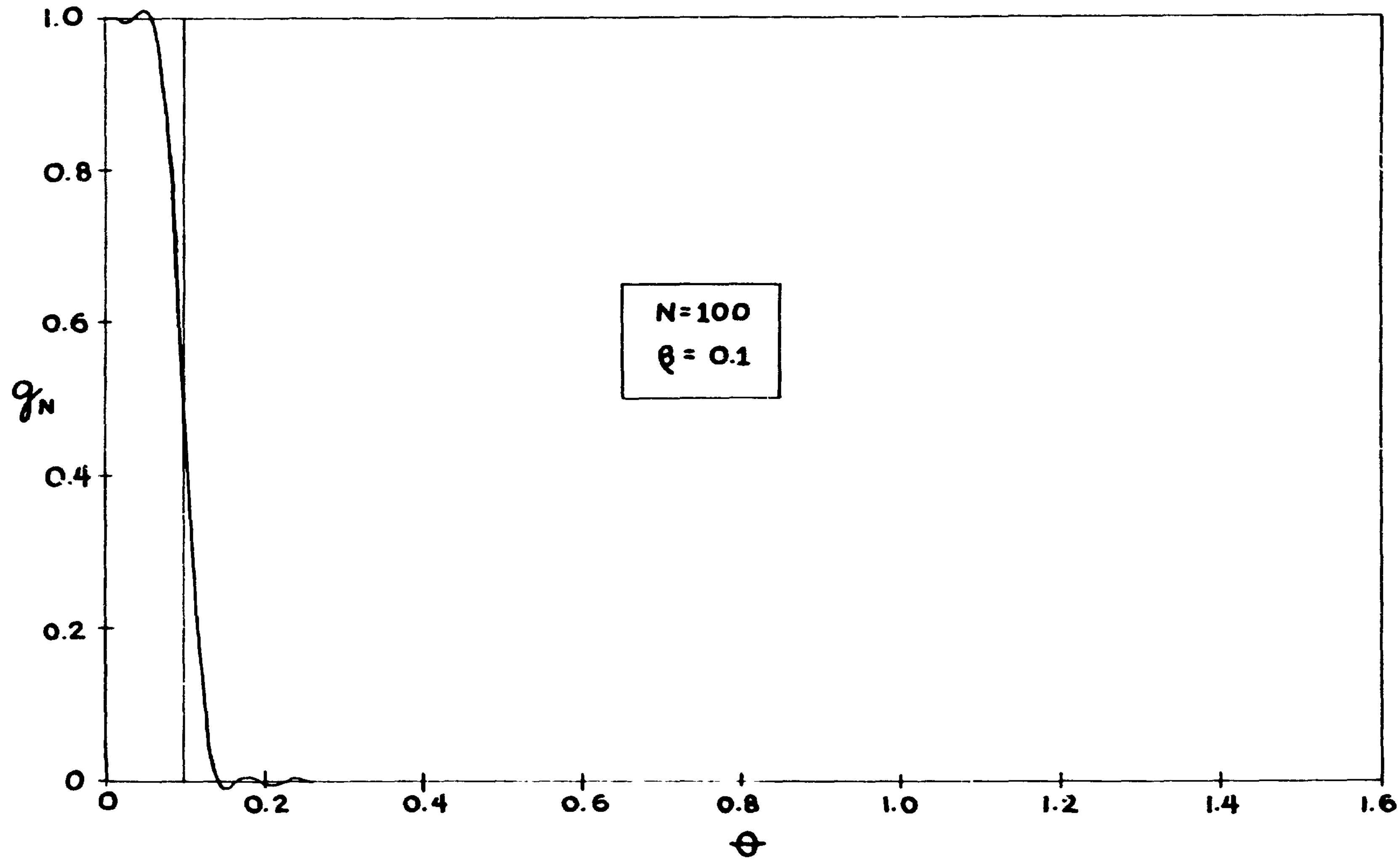

FIGURE 2. LOAD DISTRIBUTION FUNCTION VS. POLAR ANGLE

The solution of this example by the shell theories is given in Appendix III.

Next the angular extent of the load (β) and the number of terms (N) used in the load representation must be chosen. In order to detect all the peculiarities of each theory and to emphasize the possible differences between the theories it is desirable to choose β as small as possible. However, as β is decreased the number of terms (N) necessary to obtain a reasonable approximation of the load increases very rapidly. Therefore some compromise must be made to obtain a sufficiently concentrated load representable by a reasonable number of terms. After studying several different load representations it was decided to choose $\beta = \pi - 0.1$ with $N = 100$. The load distribution function $g_N(\theta)$ is shown in Figure (2) for these parameters.

Having specified all the load and shell parameters we may now proceed to the actual computation of the solution. First the natural frequencies must be found for each theory. For the shell theories this is a straightforward computation which involves finding the roots of a quadratic polynomial for the Flügge theory and a cubic polynomial for the improved theory, for each harmonic n. However, for the elasticity theory this involves finding the roots of the transcendental equation $D_n(\omega) = 0$ for each harmonic n. If approximate values for these roots are known they may be used as starting values for a first order Newton-Raphson iteration scheme to obtain the natural frequencies to any desired accuracy. The asymptotic relations (58) may be used in this procedure as follows. For each value of n, $D_n(\omega)$ is plotted for $\omega > 0$ until it is noted that the roots are obeying the asymptotic formalae (58). The starting values for the lowest roots are then obtained from the plot and the starting values for the remaining roots are obtained from the asymptotic formulae. Table (1) lists some of the starting values for the first few harmonics and indicates when the asymptotic relations become valid by using the appropriate asymptotic formula from (58) rather than its numerical value.

TABLE I

ω_{nj}

STARTING VALUES

n \ j	0	1	2	3	4	5	6
0	0.90	$\frac{\pi}{\alpha}$	$2\frac{\pi}{\alpha}$	$3\frac{\pi}{\alpha}$	$4\frac{\pi}{\alpha}$	$5\frac{\pi}{\alpha}$	$6\frac{\pi}{\alpha}$
1	1.27	$\gamma\frac{\pi}{\alpha}$	$\frac{\pi}{\alpha}$	$2\gamma\frac{\pi}{\alpha}$	$3\gamma\frac{\pi}{\alpha}$	$2\frac{\pi}{\alpha}$	$4\gamma\frac{\pi}{\alpha}$
2	0.06	2.01	$\gamma\frac{\pi}{\alpha}$	$\frac{\pi}{\alpha}$	$2\gamma\frac{\pi}{\alpha}$	$3\gamma\frac{\pi}{\alpha}$	$2\frac{\pi}{\alpha}$
3	0.19	2.84	$\gamma\frac{\pi}{\alpha}$	$\frac{\pi}{\alpha}$	$2\gamma\frac{\pi}{\alpha}$	$3\gamma\frac{\pi}{\alpha}$	$2\frac{\pi}{\alpha}$
4	0.37	3.70	$\gamma\frac{\pi}{\alpha}$	$\frac{\pi}{\alpha}$	$2\gamma\frac{\pi}{\alpha}$	$3\gamma\frac{\pi}{\alpha}$	$2\frac{\pi}{\alpha}$

All the larger values of n follow the pattern set by $n = 2, 3$ and 4, however the value of j at which the pattern becomes evident increases as n increases. This presents no difficulties since the starting values for $n = 6, 8, 10, \ldots, 100$ may be computed from a difference scheme as follows.

$$\omega_{nj} = \omega_{n-2,j} + (\omega_{n-2,j} - \omega_{n-4,j})$$

In this scheme $\omega_{n-2,j}$ and $\omega_{n-4,j}$ are the correct frequencies obtained from previous calculations and ω_{nj} is the starting value for the Newton-Raphson iteration. The difference scheme was formulated to compute only the even harmonics since the odd harmonics are not used in the present example. Using this procedure to generate the starting values, the natural frequencies were obtained to eight significant figures with usually no more than three iterations.

Note that the $j = 0$ root for $n = 1$ seems to belong with the $j = 1$ group of roots for the higher harmonics $n = 2, 3, 4$. This occurs because the lowest frequency for $n = 1$ is $\omega = 0$ which gives rise to the rigid body motions already contained in $W_1^{(s)}$ and $V_1^{(s)}$ equations (48-a, b). Therefore it is absent from this Table and it is not a root of $D_1(\omega) = 0$.

TABLE II
COMPARISON OF NATURAL FREQUENCIES

N	J	Elasticity Theory	Improved Theory	Flügge Theory
0	0	.90453	.90388	.90388
0	1	31.43019		
0	2	62.83898		
0	3	94.25253		
0	4	125.66727		
0	5	157.08248		
0	6	188.49793		
2	0	.06964	.06957	.06997
2	1	2.01483	2.01519	2.02061
2	2	16.94201	17.33108	
2	3	31.26898		
2	4	33.81500		
2	5	50.38530		
2	6	62.87608		
4	0	.37019	.36970	.37942
4	1	3.70998	3.71459	3.72648
4	2	17.28616	17.69410	
4	3	30.90754		
4	4	34.33510		
4	5	50.37718		
4	6	62.98677		
6	0	.85090	.84940	.90011
6	1	5.46350	5.47943	5.49793
6	2	17.83391	18.27320	
6	3	30.49687		
6	4	35.01135		
6	5	50.36573		
6	6	63.16919		
10	0	2.22634	2.21974	2.56819
10	1	8.97353	9.05096	9.08409
10	2	19.41665	19.95633	
10	3	29.76484		
10	4	36.60387		
10	5	50.34532		
10	6	63.73688		
20	0	6.93539	6.89078	10.38735
20	1	17.21416	18.01753	18.10386
20	2	24.97077	26.03038	
20	3	29.37554		
20	4	41.33374		
20	5	50.51445		
20	6	66.13638		
50	0	22.93459	22.68683	45.16705
50	1	29.62303	44.74099	65.19103
50	2	42.05902	49.85375	
50	3	48.12222		
50	4	57.86813		
50	5	58.32040		
50	6	73.89630		
100	0	48.01206	48.37416	90.35017
100	1	51.48581	88.61057	260.79596
100	2	61.83290	94.47102	
100	3	73.38704		
100	4	84.79185		
100	5	93.98815		
100	6	102.16473		

A partial list of the natural frequencies predicted by both the elasticity theory and the shell theories is given in Table (2). Only even harmonics are shown since the odd harmonics are not used in the example. Note that the lowest frequency is $\omega_{2,0}$ indicating that the shell offers the least resistance to motion in the n = 2 mode. Since the n = 2 mode is also prominant in the load representation we expect this to be the dominant mode of response for the shell in this example. The period of the response in this mode is $T = \frac{2\pi}{\omega_{2,0}} \cong 90$ dimensionless units of time. Next observe that the frequencies predicted by the Flügge theory are consistently greater than the corresponding frequencies predicted by either the improved or elasticity theories with the exception of n = 0. For n = 2 the lowest frequency predicted by the Flügge theory is only 0.475% greater than the lowest frequency predicted by the elasticity theory, however, this difference increases as n increases so that at n = 100 it is 87.5%. Therefore, the Flügge theory should satisfactorily predict the response caused by the low harmonic components of the load. This characteristic will rapidly deteriorate as n increases. On the other hand, the lowest frequency predicted by the improved theory is always within 1% of the corresponding frequency predicted by the elasticity theory for the range of harmonics covered by this Table. Therefore, we may expect the improved theory to satisfactorily predict the response caused by all the harmonics in this example. One of the most obvious differences between the theories is the number of frequencies associated with each theory for the various harmonics. In the elasticity theory a complete set of radial eigenfunctions and associate eigenvalues (frequencies) is required to represent the proper radial variation of the response. The shell theories on the other hand, only represent the gross effects of these radial variations for any position θ on the shell. Since for each harmonic, except n = 0, the response is due to the combined effects of shear and dilatation, the elasticity theory contains two sets of frequencies, one associated with the

dilatational effect ($\omega_{n1}, \ldots, \frac{K\pi}{\alpha}, \ldots$) and one associated with the shear effect ($\omega_{no}, \ldots, \gamma\frac{L\pi}{\alpha}, \ldots$). The Flügge theory approximates only the lowest frequency from each set (ω_{no}, ω_{n1}) while the improved theory includes also the second frequency associated with the shear effect (ω_{no}, ω_{n1}, ω_{n2}).

When computing the response predicted by the elasticity theory for each harmonic, the series will necessarily be truncated after summing a finite number of terms. From (59) we see that at the boundaries ($r = X, Y$) all the series converge uniformly for all t. However, for $Y < r < X$ there are step functions or discontinuities in the radial and hoop stresses which periodically recur with period $t = 2\alpha$ at any given value of r. To accurately represent these step functions the series will be summed with the Lanczos smoothing factor up to and including the $K = 100$ term. This corresponds to suming $0 \leq j \leq 100$ for $n = 0$ and $0 \leq j \leq 288$ for $n \geq 2$.

One further interesting feature of the response may be deduced from this Table of frequencies. It may be shown [34] that for improved theory for large n

$$\omega_{n1} \sim 0.496n$$
$$\omega_{n2} \sim 0.878n$$
$$\omega_{n3} \sim 0.929n$$

while for Flügge theory

$$\omega_{n1} \sim 0.0261n^2$$
$$\omega_{n2} \sim 0.9035n$$

If we simultaneously examine the angular and time dependence of the response for any value of j it is of the form $\cos \omega_{nj}t \, \cos n\theta$ which may be written as $\cos n(\theta \pm \frac{\omega_{nj}}{n} t)$. In the improved theory for any value of j, $\frac{\omega_{nj}}{n} \approx V_j$ where V_j is a constant so that the solution is a sum of harmonic components of the form $\cos n(\theta \pm Vt)$. This represents a wave propagating around the circumference of the sheil with phase velocity V. Thus it is possible in the improved theory to observe wave phenomena related to the angular coordinate but not to the radial

coordinate. In the Flügge theory $\frac{\omega_{n1}}{n} \sim .0261n$ so that the harmonic components of the response are now in the form $\cos n(\theta \pm .0261nt)$. Since the phase of each component varies with n, the high frequency components will be out of phase with each other and a traveling wave will not be observed.

The elastiticy theory frequencies listed in Table (2) were checked, when possible, with those given by Armenakas, Gazis and Herrmann [19] and complete agreement was found. The asymptotic relations [58] were derived earlier by Gazis [7] and [11] and further discussion of free vibration characteristics may be found in references [6] through [19].

We now have all the information necessary to quantitatively compute the response of the shell predicted by each of the three theories. The results of this computation are presented in Figures (3) through (15). The comparison of the theories is shown for the radial and circumferential displacements at the median surface of the shell ($r = 1$) and for the hoop stress at the inner and outer surfaces ($r = 0.95, 1.05$). The radial and shear stresses predicted by elasticity theory at the median surface ($r = 1$) have also been computed.

A comparison of the static solutions is shown in Figures (3), (4), (5) and (6). Note the excellent agreement between the improved and elasticity theories. At their maximum values the improved and elasticity theories differ by less than half a percent in these results, however, the Flügge and elasticity theories differ by five to six percent. The maximum static radial displacement occurs at $\theta = 0$. The maximum static tangential displacement occurs at $\theta = 0.76775$ and the maximum static hoop stress occurs at $\theta = 0$, $r = 0.95$. At any given value of r and θ, the maximum value of the dynamic response is expected to be twice the static value (see [4], pp. 181-182). Therefore, the displacements and the hoop stress predicted by the three theories in the dynamic case will be compared at the values of r and θ given above.

The static shear and radial stresses predicted by the elasticity theory at the median surface ($r = 1.00$) are shown in Figures (7) and (8). The maximum stress occurs in both cases near the edge of the load at $\theta = 0.14715$ as expected. The maximum magnitude of the hoop stress is seen to be twenty times greater than the maximum shear stress and forty times greater than the maximum radial stress. This provides justification for the usual assumption made is shell theory: the radial stress may be neglected compared to the hoop stress. We will now proceed to examine the dynamic response of the shell.

Figures (9) through (12) depict the initial response of the shell. In this very early stage of the response the various stress waves may be observed as they propagate through the shell and are reflected from its boundaries. The time history of the radial stress at the median surface directly beneath the load is shown in Figure (9). The events depicted on this graph may be explained as follows. At $t = 0$ the radial stress at the outer surface is discontinuously changed from zero to minus one. This discontinuity in the radial stress propagates as a compression wave into the shell with unit dimensionless velocity (dilatational wave speed). At $t = \frac{\tau}{2} = .05$ we observe this compression wave as it passes the median surface. At $t = \tau$ it encounters the inner surface of the shell and since this surface is stress free it is reflected as a tensile wave which is observed as it passes the median surface at $t = \frac{3\tau}{2} = 0.15$. At $t = 2\tau$ this tensile wave encounters the outer boundary from which it reflects as a compression wave. This phenomenon is repeated periodically with period $T = 2\tau$. The time history of the stress at the median surface becomes more complicated as each wave passes and adds its effect to those of the previous waves. This is the reason for the changing form of the response curve in Figure (9).

Figure (10) shows the shear stress predicted by elasticity theory at the median surface of the shell and at the edge of the applied load. As predicted

earlier, there are no discontinuities in the shear stress, however, the various waves may still be detected since they cause discontinuities in the slope of the curve. At $t = 0.05$ we observe the dilatational wave and then at $t = \frac{0.05}{\gamma} \cong 0.935$ we observe the shear wave as both pass the median surface. Then again at $t = 0.15$ and $t \cong 0.187$ the dilatational and shear waves reflected from the inner surtace are observed.

Figure (11) depicts the time history of the hoop stress at the median surface directly beneath the load at $\theta = 0$. A comparison of the three theories is presented on this plot. In the elasticity theory the dilatational wave is observed as it is reflected between the shell boundaries. As noted earlier this wave cannot be predicted by the shell theories; however, they do accurately characterize the average value of the stress.

Figure (12) shows the hoop stress predicted by the three theories at the inner surface of the shell at $\theta = \frac{\pi}{2}$. Because of the wave character of the elasticity theory, no response is observed until the dilatational wave originating at the edge of the load reaches this location on the shell. This time interval is approximately $t = \frac{\pi}{2} - \beta = 1.47$. Similarly the improved theory contains waves traveling around the shell with phase velocities $V_1 = 0.496$, $V_2 = 0.878$ and $V_3 = 0.929$. Therefore, there is no response observed until the fastest wave V_3 passes this point on the shell. This occurs at $t = \frac{\frac{\pi}{2} - \beta}{V_3} \cong 1.58$. As indicated previously the Flügge theory response is not entirely composed of traveling waves and therefore it predicts an immediate response at every point on the shell.

Figures (13) through (15) show one full period of the response predicted by the three theories. The radial and circumferential displacements of the median surface of the shell are shown in Figures (13) and (14). Note the excellent agreement between the improved and elasticity theories. In both cases they differ by less than half a percent in the vicinity of the maximum. The

Flügge theory however, differs from the elasticity theory by approximately seven percent in this vicinity. The static solution predicted by the elasticity theory is also shown on these graphs and, as expected, it is approximately one half of the maximum value of the dynamic response. Also, the period of the motion is seen to be T = 90 as predicted.

Figure (15) shows the hoop stress at the inner surface directly beneath the load predicted by the three theories. Again the agreement between the elasticity and improved theories is excellent both in magnitude and form. Since the response predicted by the Flügge theory is slightly out of phase with the other two theories (the Flügge theory predicts a slightly faster response) there are large differences between the theories at any given instant of time, however, the maximum value predicted by the Flügge theory differs by only nine percent from the maximum predicted by the elasticity theory.

The radial and shear stresses were computed for one complete period of the response and their maximum magnitude was found to be, as in the static case, less than four percent of the maximum hoop stress. Thus the assumption made in shell theory; i.e., the radial stress is negligible compared to the hoop stress, is also valid in the dynamic case.

This concludes the study of the specific example. The principal findings of this investigation will now be summarized in the conclusion.

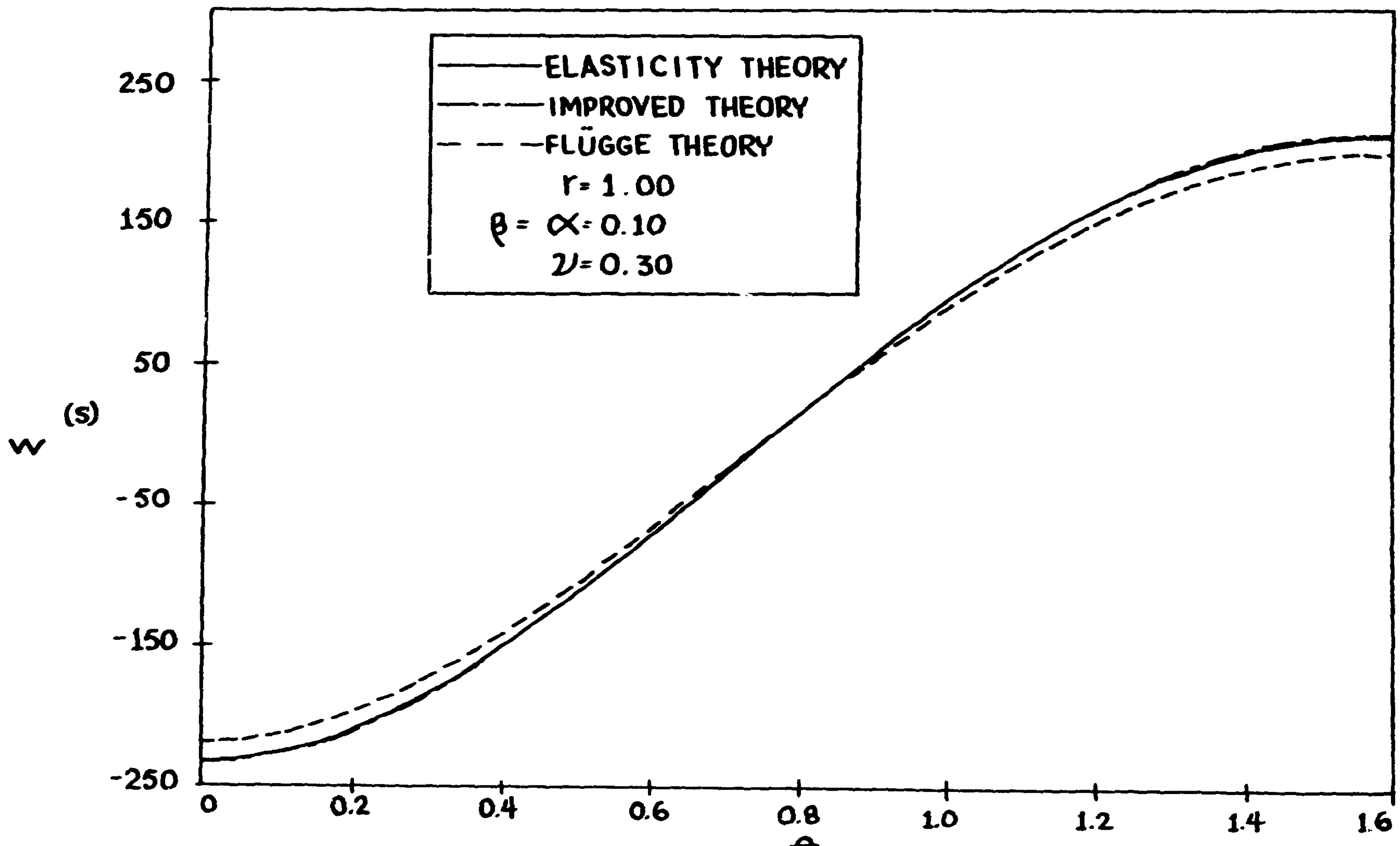

FIGURE 3. STATIC RADIAL DISPLACEMENT VS. POLAR ANGLE

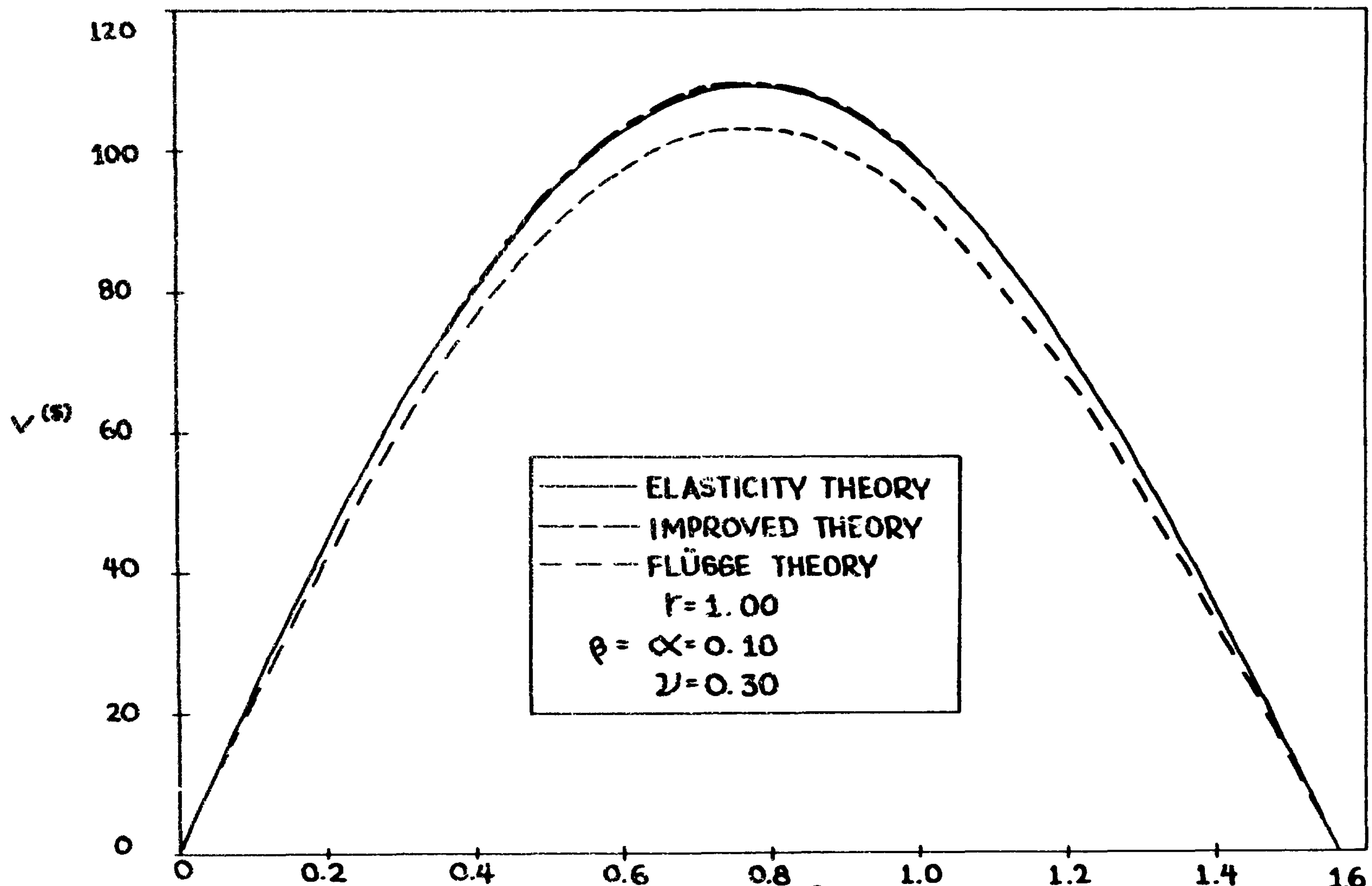

FIGURE 4. STATIC CIRCUMFERENTIAL DISPLACEMENT VS. POLAR ANGLE

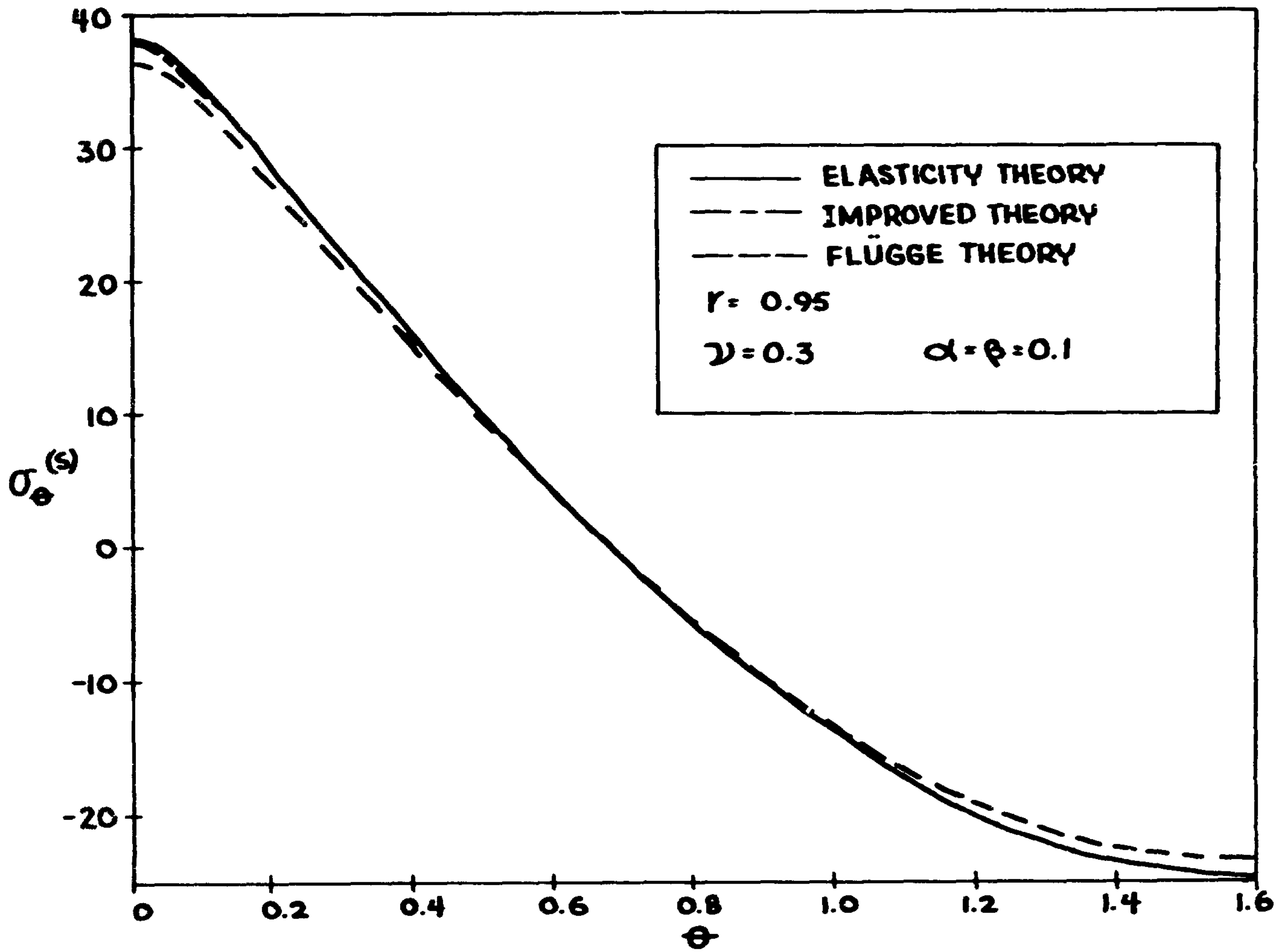

FIGURE 5. STATIC HOOP STRESS VS. POLAR ANGLE

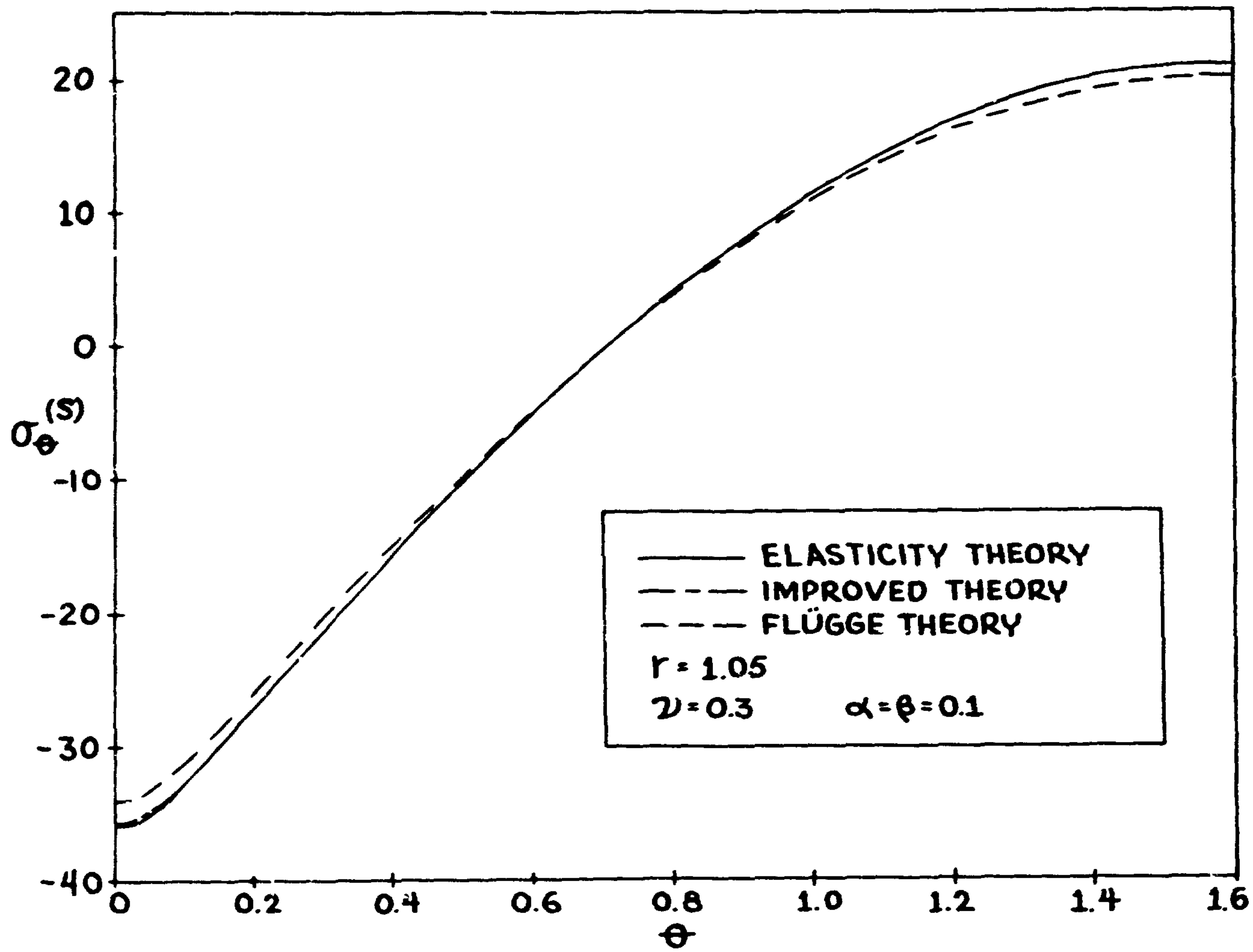

FIGURE 6. STATIC HOOP STRESS VS. POLAR ANGLE

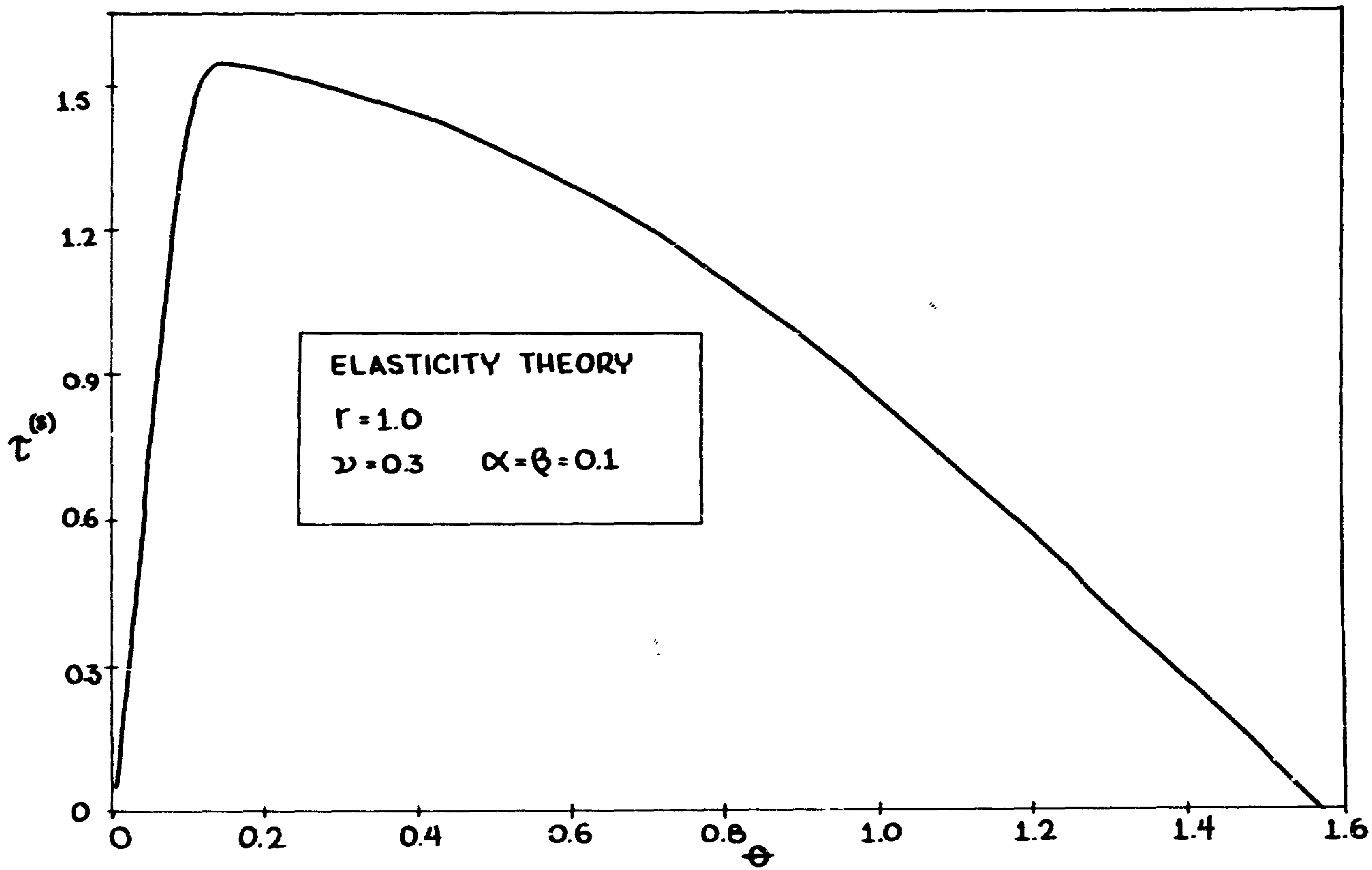

FIGURE 7. STATIC SHEAR STRESS VS. POLAR ANGLE

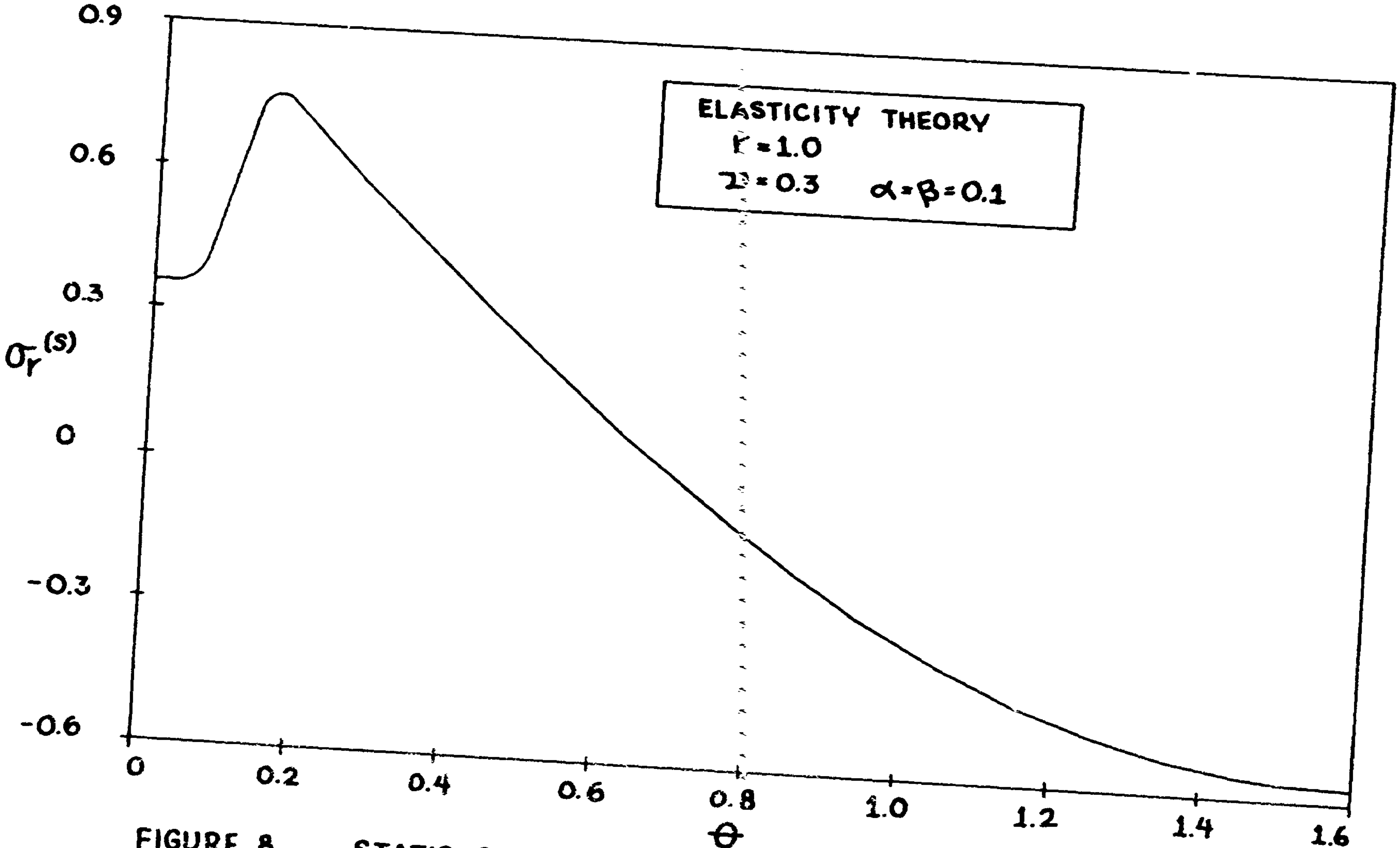

FIGURE 8. STATIC RADIAL STRESS VS. POLAR ANGLE

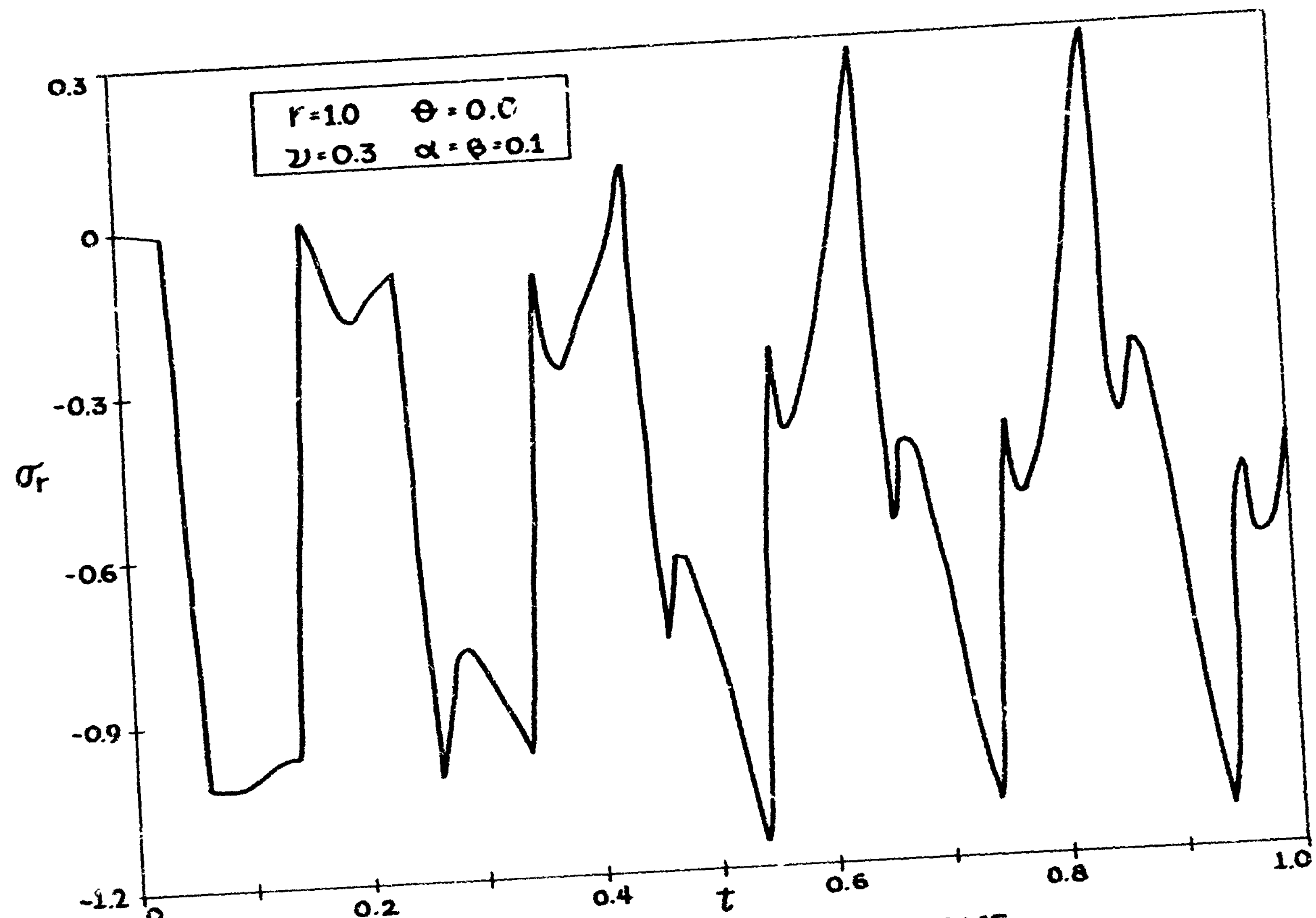

FIGURE 9. INITIAL RESPONSE: RADIAL STRESS VS. TIME

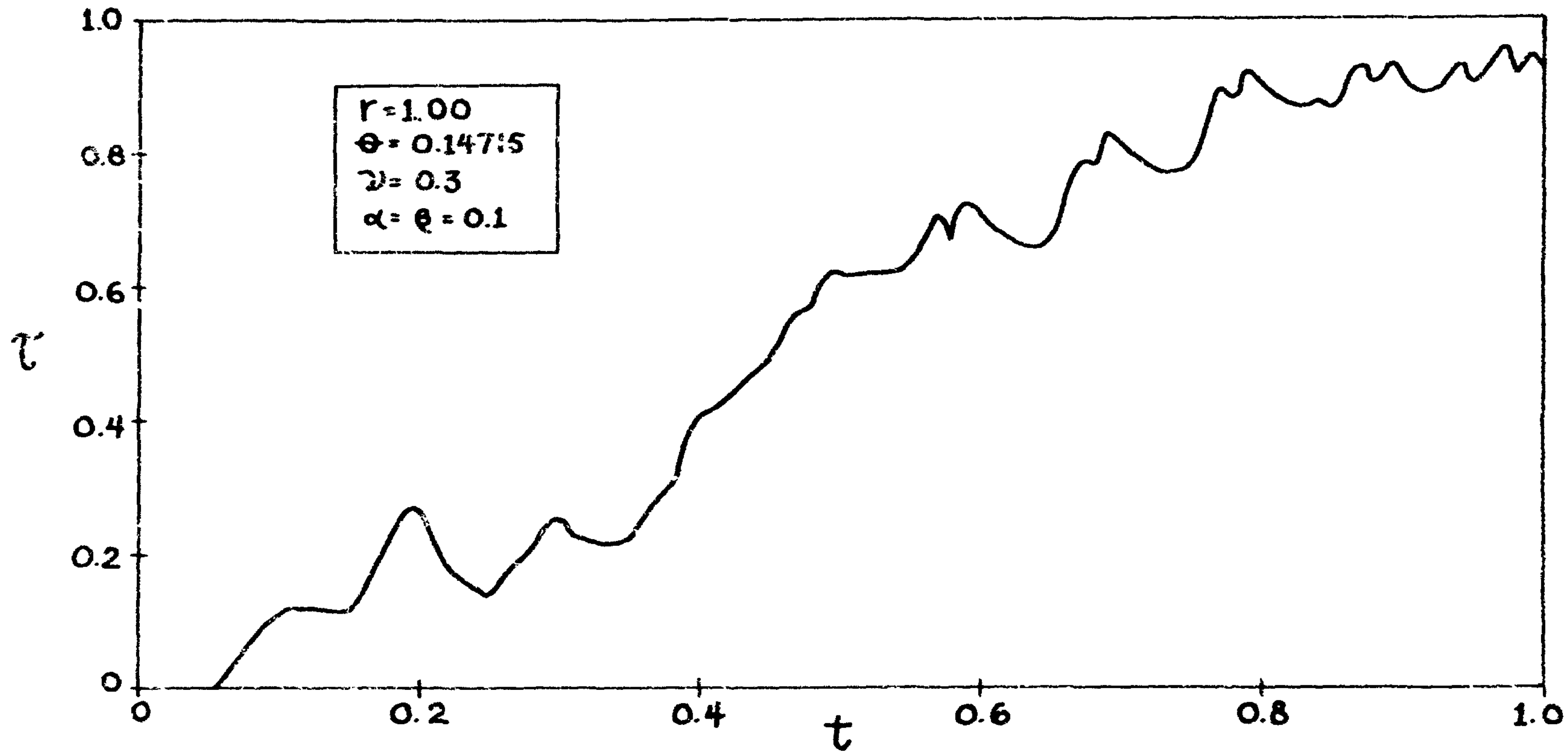

FIGURE 10. INITIAL RESPONSE: SHEAR STRESS VS. TIME

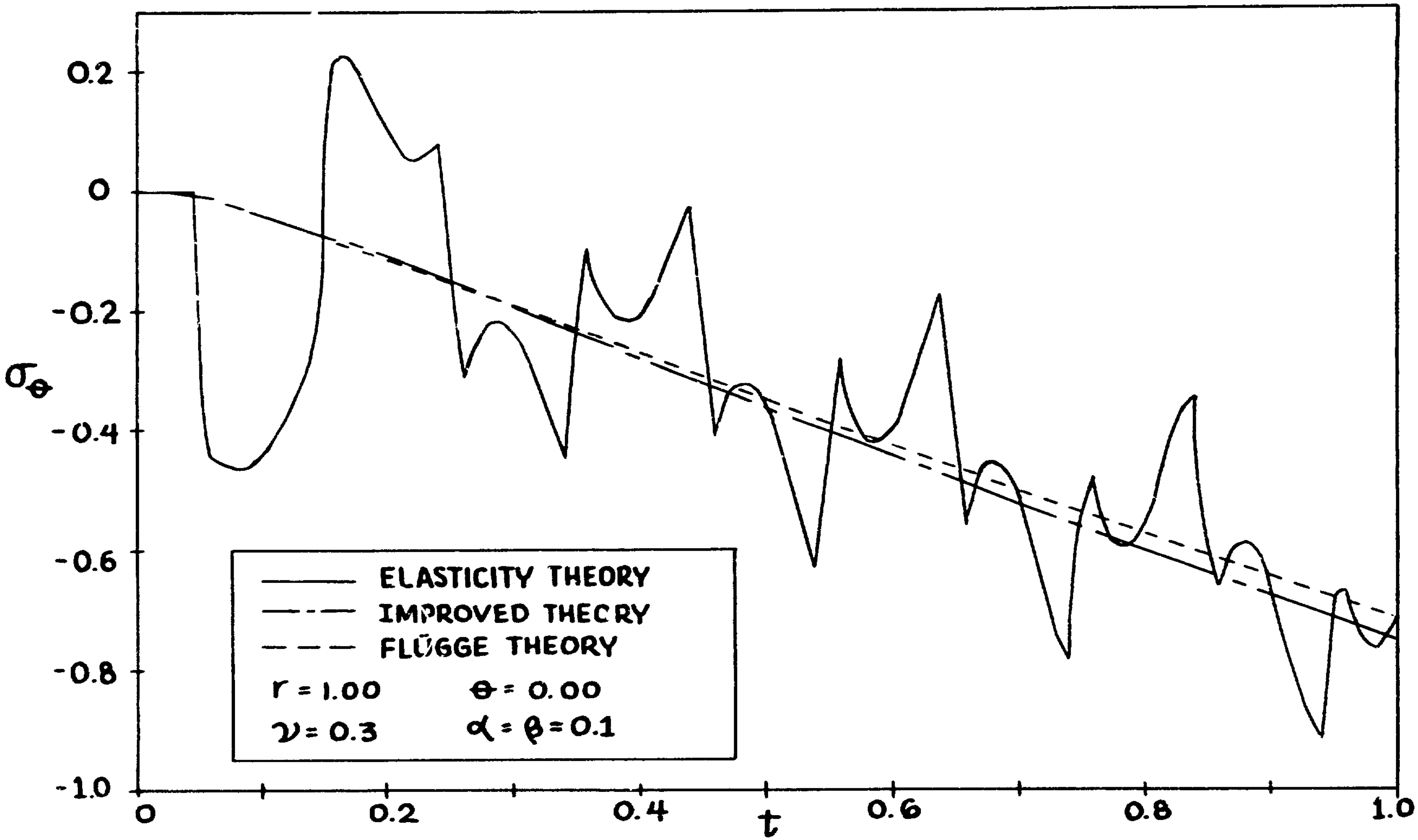

FIGURE 11. INITIAL RESPONSE: HOOP STRESS VS. TIME

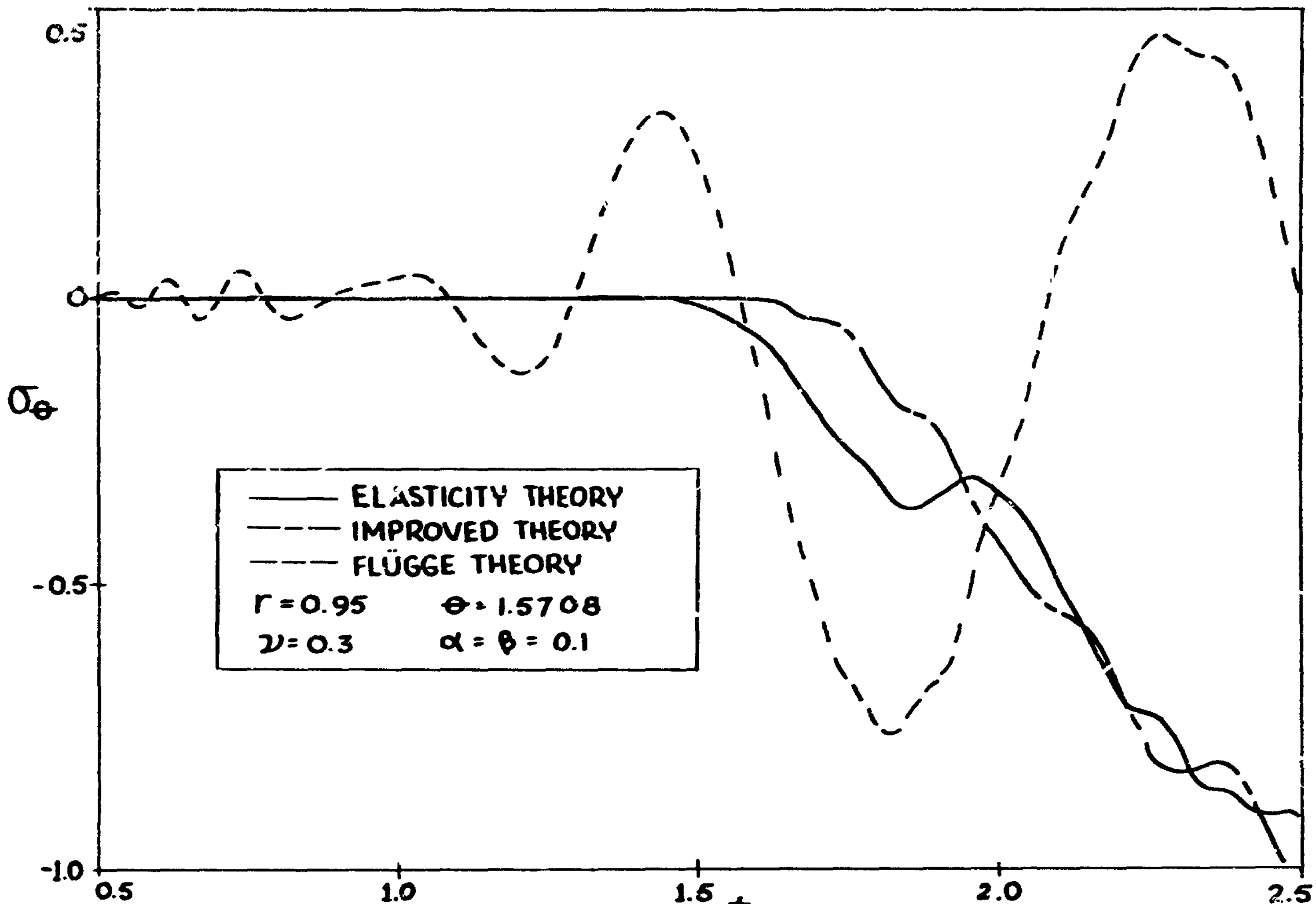

FIGURE 12. INITIAL RESPONSE: HOOP STRESS VS. TIME

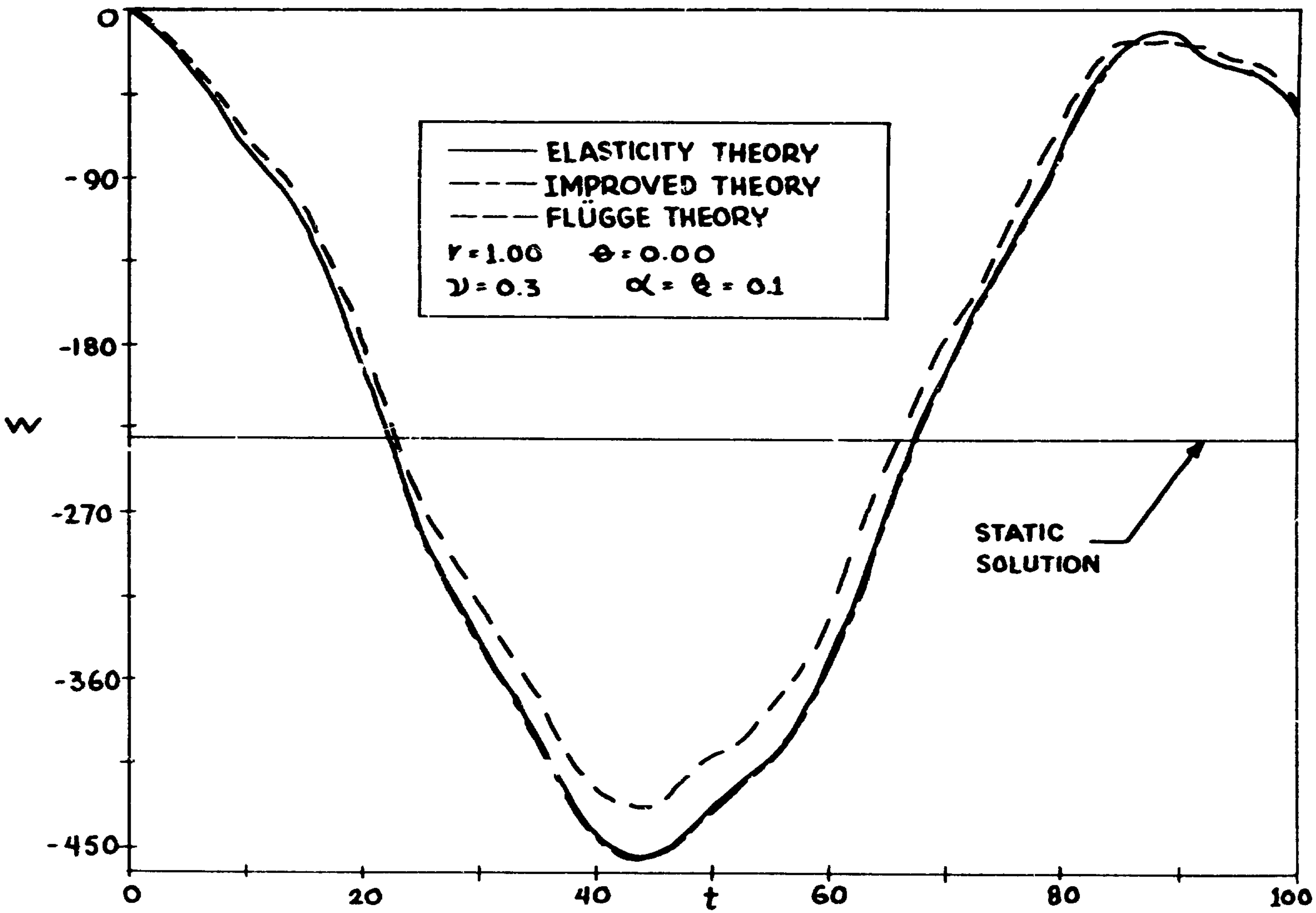

FIGURE 13. RADIAL DISPLACEMENT VS. TIME

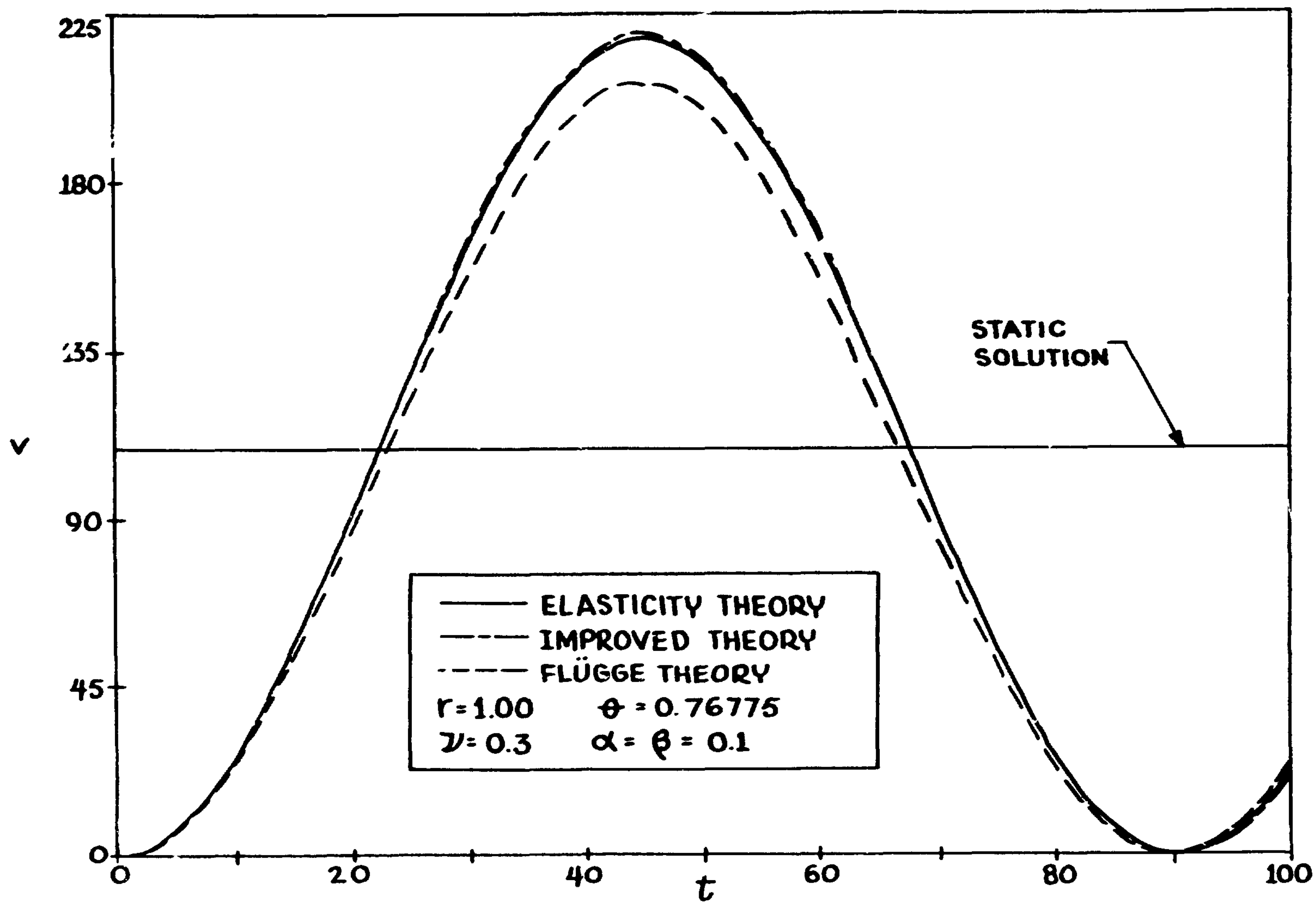

FIGURE 14. CIRCUMFERENTIAL DISPLACEMENT VS. TIME

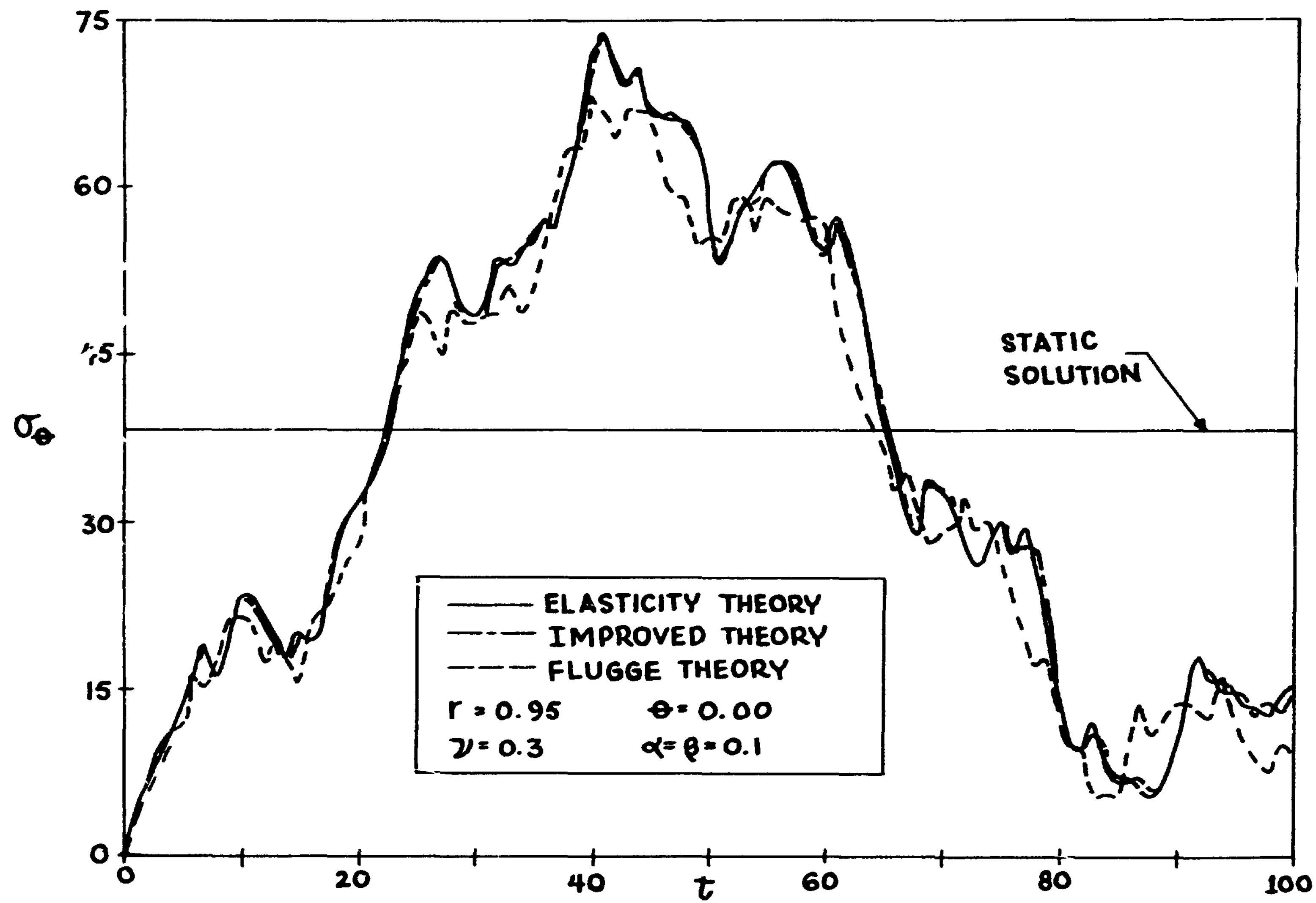

FIGURE 15. HOOP STRESS VS. TIME

V. CONCLUSIONS

The principal findings of this investigation will now be summarized. There are three convenient stages of the response for which the shell and elasticity theories have been compared. First, there is an initial response in which the dilatational and shear waves transmit the effects of the loading to the various points in the shell and the radial variation of the response begins to develop. Neither of the shell theories can accurately describe the details of the response during this early period and the elasticity theory must be employed to observe this phenomenon. The period of the wave phenomenon occurring during this initial response is $T = 2\tau$ which is the time required for a dilatational wave to travel from the outer surface to the inner surface and then back again to the outer surface. The corresponding period for the shear wave is $T = \frac{2\tau}{\gamma}$.

The second stage of the response consists of the effects of the load being transmitted around the shell in the circumferential direction. The characteristic periods of the response produced by the dilatational and shear waves in this stage are approximately $T = 2\pi$ and $T = \frac{2\pi}{\gamma}$. As shown, the improved theory does provide a good approximation of this aspect of the response with the periods being slightly larger than those predicted by elasticity theory. However, the Flügge theory cannot characterize this aspect of the response because of the nature of its frequency spectrum.

The third stage of the response contains the maximum displacements and stresses occurring in the shell for all time. One complete period of the response occurred in the interval $T = \frac{2\pi}{\omega_{2,0}}$ where $\omega_{2,0}$ was the lowest frequency predicted by the various theories. As explained earlier the $n = 2$ mode is the principal

harmonic component of the response. Since the Flügge theory very accurately characterizes this mode of response, the results predicted by the Flügge theory are surprisingly good. The inaccuracy in the Flügge theory characterization of the higher modes contributed to a total error of less than 10% in the maximum hoop stress and even less in the maximum displacements. However, the improved theory is still far superior to the Flügge theory because of its accurate characterization of the higher modes. Thus the error in the improved theory was less than 1/2% in both the maximum stress and displacement. Also it was seen that the Flügge theory always underestimated the maximum stress whereas the improved theory provided a slightly conservative estimate.

If the applied load is modified in such a way that the relative importance of the higher modes is decreased the Flügge theory may be expected to yield an even better approximation of the overall response. This would occur if the load were applied at a finite rate, if the angular extent of the load were increased or if the load were distributed continuously over the shell surface. On the other hand if the load becomes more concentrated the error in the Flügge theory response will increase. However, since the relative difficulty of both shell theories is about equal compared to elasticity theory and since the improved theory yields a closer approximation to the actual response, the improved theory must be judged to be superior to Flügge shell theory.

One further factor to be considered when comparing the shell theories to the elasticity theory is the computation time necessary to obtain the solution. All computations for the example presented were programmed in Fortran IV on a CDC 6400 computer. The computing time required for each shell theory was approximately ten minutes, however the computing time required for the elasticity theory was approximately four hours. Elasticity theory required a greater amount of time because of the number of terms summed in the series for

each harmonic to obtain the initial response. If the initial response is not desired, considerably fewer terms may be used in these series. However, the computing time will still greatly exceed that required by the shell theories because the Bessel functions involved in the solution require more computing time than the algebraic functions involved in the shell theory solution. Thus if the initial response is not desired, this consideration represents an additional advantage of improved shell theory.

In conclusion, it appears that improved shell theory is the superior choice of the three theories in terms of the trade-off between accuracy and analytical complexity if the details of the initial response are not required. If the initial response is desired elasticity theory must be used. We also note that there are several shell theories which are similar to Flügge's theory. Thus it may be expected that the present findings with respect to Flügge's theory are also applicable to the shell theories of Love, Donnell, Vlasov, Sanders, etc.

APPENDIX I: LANCZOS' SMOOTHING TECHNIQUE

The accuracy of the approximation of a function over some interval by a truncated Fourier series may usually be increased by applying the operation of local smoothing to the function being represented. This procedure along with some illustrative examples are presented by Lanczos in his book on Fourier series ([38], pp. 61-75). The crux of the method is the replacement of the original function by one which is locally smooth. For example, suppose we wish to represent the function f(X) defined over the interval $-L < X < L$ by a Fourier series which has been truncated after the K^{th} term in the series. Then

$$f_K(X) = \frac{a_o}{2} + \sum_{j=1}^{K} (a_j \cos \omega_j X - b_j \sin \omega_j X)$$

where a_o, a_j and b_j are the Fourier coefficients of f(X) and $\omega_j = \frac{j\pi}{L}$. Consider a new function $\bar{f}(X)$, derived from f(X) as follows.

$$\bar{f}(X) = \frac{\omega_K}{2\pi} \int_{-\frac{\pi}{\omega_K}}^{\frac{\pi}{\omega_K}} f(X+Y)dY$$

The following conclusions may be obtained through the application of the mean value theorem to the above integral. If f is continuous at X_o, then $\lim_{X \to X_o} [\lim_{K \to \infty} \bar{f}(X)] = f(X_o)$. If f is discontinuous at X_o, then $\lim_{X \to X_o} [\lim_{K \to \infty} \bar{f}(X)] = \frac{f^+ + f^-}{2}$ where $f^+ = \lim_{\epsilon \to o} f(X_o + \epsilon)$ and $f^- = \lim_{\epsilon \to o} f(X_o - \epsilon)$. For large but finite values of K, this smoothing procedure has very little effect on $f(X_o)$ if f is continuous over the interval $(X_o - \frac{\pi}{\omega_K} < X < X_o + \frac{\pi}{\omega_K})$. However, if f has a discontinuity at X_o,

the discontinuity will be smoothed into a rapid but continuous change. The truncated Fourier series approximation of this smoothed function may be obtained by substituting f_K into the integral definition to obtain.

$$\bar{f}_K(X) = \frac{a_o}{2} + \sum_{j=1}^{K} \frac{\omega_K}{\omega_j \pi} \sin \frac{\omega_j \pi}{\omega_K} (a_j \cos \omega_j X - b_j \sin \omega_j X)$$

The accuracy of this approximation of f(X) is better than the truncated Fourier series at all points except those in the immediate neighborhood of a discontinuity.

APPENDIX II: CROSS PRODUCTS OF BESSEL FUNCTIONS

The properties of the cross products of the Bessel functions defined in equations (28) and (42) are listed below.

For $0 < X < \infty$, $0 < Y < \infty$, $K = (1,2,3,4)$ and $n = 0,1,2,\ldots$

(1) $\bar{F}_n^{(K)}(P,X,Y)$ are entire functions of P

(2) $\bar{F}_n^{(K)}(-P,X,Y) = \bar{F}_n^{(K)}(P,X,Y)$

(3) $\bar{F}_n^{(K)}(i\omega,X,Y) = -\frac{\pi}{2} F_n^{(K)}(\omega,X,Y)$; $(K,n) \neq (4,0)$

$$\bar{F}_o^{(4)}(i\omega,X,Y) = \frac{\pi}{2} F_o^{(4)}(\omega,X,Y)$$

where $i = \sqrt{-1}$ and ω is real.

(4) $\lim\limits_{P \to o} \bar{F}_n^{(1)}(P,X,Y) = \frac{1}{2n}\left[\left(\frac{X}{Y}\right)^n - \left(\frac{Y}{X}\right)^n\right]$

$$\lim_{P \to o} \bar{F}_n^{(2)}(P,X,Y) = -\frac{1}{2}\left[\left(\frac{X}{Y}\right)^n + \left(\frac{Y}{X}\right)^n\right]$$

$$\lim_{P \to o} \bar{F}_n^{(3)}(P,X,Y) = \frac{1}{2}\left[\left(\frac{X}{Y}\right)^n + \left(\frac{Y}{X}\right)^n\right]$$

$$\lim_{P \to o} \bar{F}_n^{(4)}(P,X,Y) = -\frac{n}{2}\left[\left(\frac{X}{Y}\right)^n - \left(\frac{Y}{X}\right)^n\right], \; n \neq 0$$

$$\lim_{P \to o} \bar{F}_o^{(4)}(P,X,Y) = -\frac{1}{2}\left[\frac{X}{Y} - \frac{Y}{X}\right]$$

(5) As $P \to \infty$ the functions behave asymptotically as follows.

$$\bar{F}_n^{(1)}(P,X,Y) \sim \frac{1}{\sqrt{XY}} \frac{1}{P} \sinh P(X-Y)$$

$$\bar{F}_n^{(2)}(P,X,Y) \sim -\sqrt{\frac{Y}{X}} \cosh P(X-Y)$$

$$\bar{F}_n^{(3)}(P,X,Y) \sim \sqrt{\frac{X}{Y}}\ \cosh P(X-Y)$$

$$\bar{F}_n^{(4)}(P,X,Y) \sim \sqrt{XY}\ P \sinh P(X-Y) \quad ; n \neq 0$$

$$\bar{F}_o^{(4)}(P,X,Y) \sim \frac{-1}{\sqrt{XY}}\ \frac{1}{P} \sinh P(X-Y)$$

(6)

(a) For n = 0

$$\frac{\partial}{\partial \omega} F_o^{(1)}(\omega,X,Y) = \frac{1}{\omega}\left[F_o^{(2)}(\omega,X,Y) + F_o^{(3)}(\omega,X,Y)\right]$$

$$\frac{\partial}{\partial \omega} F_o^{(2)}(\omega,X,Y) = \omega Y\left[XF_o^{(4)}(\omega,X,Y) - YF_o^{(1)}(\omega,X,Y)\right]$$

$$\frac{\partial}{\partial \omega} F_o^{(3)}(\omega,X,Y) = \omega X\left[YF_o^{(4)}(\omega,X,Y) - XF_o^{(1)}(\omega,X,Y)\right]$$

$$\frac{\partial}{\partial \omega} F_o^{(4)}(\omega,X,Y) = \frac{-1}{\omega}\left[\frac{X}{Y} F_o^{(2)}(\omega,X,Y) + \frac{Y}{X} F_o^{(3)}(\omega,X,Y) + 2F_o^{(4)}(\omega,X,Y)\right]$$

(b) For n = 1, 2, 3, . . .

$$\frac{\partial}{\partial \omega} F_n^{(1)}(\omega,X,Y) = \frac{1}{\omega}\left[F_n^{(2)}(\omega,X,Y) + F_n^{(3)}(\omega,X,Y)\right]$$

$$\frac{\partial}{\partial \omega} F_n^{(2)}(\omega,X,Y) = \frac{1}{\omega}\left[F_n^{(4)}(\omega,X,Y) + (n^2 - \omega^2 Y^2)F_n^{(1)}(\omega,X,Y)\right]$$

$$\frac{\partial}{\partial \omega} F_n^{(3)}(\omega,X,Y) = \frac{1}{\omega}\left[F_n^{(4)}(\omega,X,Y) + (n^2 - \omega^2 X^2)F_n^{(1)}(\omega,X,Y)\right]$$

$$\frac{\partial}{\partial \omega} F_n^{(4)}(\omega,X,Y) = \frac{-1}{\omega}\left[(\omega^2 X^2 - n^2)\ F_n^{(2)}(\omega,X,Y) + (\omega^2 Y^2 - n^2)F_n^{(3)}(\omega,X,Y)\right]$$

(7)

(a) For n = 0

$$\frac{\partial}{\partial X} F_o^{(1)}(\omega,X,Y) = \frac{1}{X} F_o^{(3)}(\omega,X,Y)$$

$$\frac{\partial}{\partial X} F_o^{(2)}(\omega,X,Y) = \omega^2 Y\ F_o^{(4)}(\omega,X,Y)$$

$$\frac{\partial}{\partial Y} F_o^{(1)}(\omega,X,Y) = \frac{1}{Y} F_o^{(2)}(\omega,X,Y)$$

$$\frac{\partial}{\partial Y} F_o^{(3)}(\omega, X, Y) = \omega^2 X F_o^{(4)}(\omega, X, Y)$$

(b) For n = 1, 2, 3, . . .

$$\frac{\partial}{\partial X} F_n^{(1)}(\omega, X, Y) = \frac{1}{X} F_n^{(3)}(\omega, X, Y)$$

$$\frac{\partial}{\partial X} F_n^{(2)}(\omega, X, Y) = \frac{1}{X} F_n^{(4)}(\omega, X, Y)$$

$$\frac{\partial}{\partial X} F_n^{(3)}(\omega, X, Y) = -\frac{1}{X}(\omega^2 X^2 - n^2) F_n^{(1)}(\omega, X, Y)$$

$$\frac{\partial}{\partial X} F_n^{(4)}(\omega, X, Y) = -\frac{1}{X}(\omega^2 X^2 - n^2) F_n^{2}(\omega, X, Y)$$

$$\frac{\partial}{\partial Y} F_n^{(1)}(\omega, X, Y) = \frac{1}{Y} F_n^{(2)}(\omega, X, Y)$$

$$\frac{\partial}{\partial Y} F_n^{(3)}(\omega, X, Y) = \frac{1}{Y} F_n^{(4)}(\omega, X, Y)$$

APPENDIX III: SHELL THEORY SOLUTIONS

The response of a cylindrical shell in plane strain produced by an arbitrary, radially directed load $P(\theta,t)$ is given in [39], p. 22 and [34], p. 297, for both the Flügge and improved shell theories. In terms of the nondimensionalization used in this study these solutions may be rewritten in the following form. For the Flügge Theory:

$$w(\theta,t) = -\frac{1}{\alpha\pi}\left[\frac{1}{2}F_o(t) + \sum_{n=1}^{N}\sum_{i=1}^{2} L_n A_{in} F_{in}(\theta,t)\right]$$

$$v(\theta,t) = -\frac{1}{\alpha\pi}\sum_{n=1}^{N}\sum_{i=1}^{2} L_n B_{in} G_{in}(\theta,t)$$

$$\tau_\theta(r,\theta,t) = \frac{1}{\alpha r\pi}\,\frac{(1-2\nu)}{(1-\nu)^2}\left[\frac{1}{2}F_o(t) + \sum_{n=1}^{N}\sum_{i=1}^{2} L_n\left(A_{in} + nB_{in}r + n^2(r-1)A_{in}\right)F_{in}(\theta,t)\right]$$

For the Improved Theory:

$$w(\theta,t) = -\frac{X}{\alpha\pi}\left[\frac{1}{2}F_o(t) + \sum_{n=1}^{N}\sum_{i=1}^{3} L_n A_{in} F_{in}(\theta,t)\right]$$

$$v(\theta,t) = -\frac{X}{\alpha\pi}\left[\sum_{n=1}^{N}\sum_{i=1}^{3} L_n B_{in} G_{in}(\theta,t)\right]$$

$$\tau_\theta(r,\theta,t) = -\frac{X}{\alpha r\pi}\,\frac{(1-2\nu)}{(1-\nu)^2}\left[\frac{1}{2}F_o(t) + \sum_{n=1}^{N}\sum_{i=1}^{3} L_n\left(A_{in} + nB_{in} + n(r-1)C_{in}\right)F_{in}(\theta,t)\right]$$

where

$$F_o(t) = \frac{1}{\omega_o} \int_o^t \int_{-\pi}^{\pi} P(\phi,\tau)\sin\omega_o(t-\tau)\,d\phi\, d\tau$$

$$F_{in}(\theta,t) = \frac{1}{\omega_{in}} \int_o^t \int_{-\pi}^{\pi} P(\phi,\tau)\cos n(\theta-\phi)\sin\omega_{in}(t-\tau)\,d\phi\, d\tau$$

$$G_{in}(\theta,t) = \frac{1}{\omega_{in}} \int_o^t \int_{-\pi}^{\pi} P(\phi,\tau)\sin n(\theta-\phi)\sin\omega_{in}(t-\tau)\,d\phi\, d\tau$$

$$L_n = \frac{N}{n\pi}\sin\frac{n\pi}{N}$$

$$\omega_{in} = \frac{\sqrt{1-2\nu}}{(1-\nu)}\,\Omega_{in} \quad , \quad \omega_o = \frac{\sqrt{1-2\nu}}{(1-\nu)}\,\Omega_o$$

For the example discussed in Section IV $P(\theta,t) = g(\theta)H(t)$

where $$g(-\theta) = g(\theta) = \begin{Bmatrix} 1 & , & 0 \le \theta < \beta \\ 0 & , & \beta < \theta < \pi-\beta \\ 1 & , & \pi-\beta < \theta \le \pi \end{Bmatrix}$$

Therefore, for this example

$$F_o(t) = 4\beta\left[\frac{1-\cos\omega_o t}{\omega_o^2}\right]$$

$$F_{in}(\theta,t) = \frac{4}{n}\sin n\beta\cos n\theta\left[\frac{1-\cos\omega_{in}t}{\omega_{in}^2}\right] ; \; n = 2,4,6,\ldots$$

$$G_{in}(\theta,t) = \frac{4}{n}\sin n\beta\sin n\theta\left[\frac{1-\cos\omega_{in}t}{\omega_{in}^2}\right] ; \; n = 2,4,6,\ldots$$

$$F_{in} = G_{in} = 0 \; ; \; n = 1,3,5,\ldots$$

The modal coefficients A_{in}, B_{in}, C_{in} and the natural frequencies Ω_{in} associated with each theory are given below. For the Flügge theory,

$$\Omega_o = \sqrt{1+\frac{\alpha^2}{12}} \; , \quad \Omega_{1n} = \sqrt{\frac{\lambda_n}{2}(1+X_n)} \; , \quad \Omega_{2n} = \sqrt{\frac{\lambda_n}{2}(1-X_n)}$$

$$A_{1n} = \frac{1}{X_n}\left(\frac{1+X_n}{2} - \frac{n^2}{\lambda_n}\right), \quad A_{2n} = \frac{-1}{X_n}\left(\frac{1-X_n}{2} - \frac{n^2}{\lambda_n}\right)$$

$$B_{1n} = \frac{n}{\lambda_n X_n}, \quad B_{2n} = -B_{1n}$$

$$\lambda_n = 1 + n^2 + \frac{\alpha^2}{12}(n^2-1)^2, \quad X_n = \sqrt{1 - \frac{\alpha^2 n^2 (n^2-1)^2}{3\lambda_n^2}}$$

For the improved theory:

$$\Omega_o = \sqrt{1 + \frac{\alpha^2}{12}}, \quad \Omega_{in} = \frac{\sqrt{12}\,\beta_{in}}{\alpha\sqrt{1 - \frac{\alpha^2}{12}}}$$

$$A_{in} = \frac{1}{d_{in}}\left[\beta_{in}^4 - a_{2n}\beta_{in}^2 + a_{in}g_{1n}\right]$$

$$B_{in} = \frac{1}{d_{in}}\left[a_{5n}\beta_{in}^2 - a_{1n}g_{1n}\right]$$

$$C_{in} = \frac{1}{d_{in}}\left[a_{6n}g_{1n} - a_{7n}\beta_{in}^2\right]$$

$$d_{in} = \left[\beta_{i+1,n}^2 - \beta_{in}^2\right]\left[\beta_{i+2,n}^2 - \beta_{in}^2\right] \text{ MOD } (3)$$

$$\beta_{1,1}^2 = \frac{g_{21}}{2}\left[1 - \sqrt{1-4g_{51}}\right]$$

$$\beta_{21}^2 = \frac{g_{21}}{2}\left[1 + \sqrt{1-4g_{51}}\right]$$

$$\beta_{31}^2 = 0$$

$$\beta_{in}^2 = \frac{1}{3}g_{2n}\left[1 - 2X_n\cos\left(\frac{n + 2(i-1)\pi}{3}\right)\right], \quad n \geq 2$$

$$X_n = \sqrt{1-3g_{5n}}$$

$$[\text{illegible}]_n = \arccos\left\{-\left[1 - \frac{9}{2}g_{5n} - \frac{27}{2}g_{6n}\right]\left[1 - 9g_{5n} + 27g_{5n}^2(1-g_{5n})\right]^{-1/2}\right\}$$

$$g_{1n} = \frac{\alpha^2}{12}\left(1 - \frac{\alpha^2}{12}\right)$$

$$g_{2n} = \frac{\alpha^2 n^2}{12}\,(2 + K^2)(1 + \frac{\lambda_1 \alpha^2}{12}) + K^2\,(1 + \frac{\lambda_2 \alpha^2}{12})$$

$$g_{3n} = \frac{\alpha^2 n^4}{12}\,(1 + 2K^2)(1 + \frac{\lambda_3 \alpha^2}{12}) + K^2 n^2\,(1 + \frac{\lambda_4 \alpha^2}{12})$$

$$+ K^2\,(1 + \frac{\alpha^2}{12})^2 (1 + \frac{\alpha^2}{4})$$

$$g_{4n} = \frac{\alpha^2 K^2}{12}\,(1 + \frac{\alpha^2}{12})\, n^2 (n^2 - 1)^2$$

$$g_{5n} = \frac{g_{1n} g_{3n}}{g_{2n}^2} \quad , \qquad g_{6n} = \frac{g_{1n}^2 g_{4n}}{g_{2n}^3}$$

$$\lambda_1 = \frac{3 - \frac{\alpha^2 K^2}{12}}{2 + K^2} \quad , \qquad \lambda_2 = 4 + \frac{\alpha^2}{4} + \frac{1 - \frac{\alpha^4}{144}}{K^2}$$

$$\lambda_3 = K^2 \left(\frac{5 + \frac{\alpha^2}{4}}{1 + 2K^2} \right) \quad , \qquad \lambda_4 = \frac{1 + K^2}{K^2} - 6 \left(1 + \frac{\alpha^2}{12} \right)^2$$

$$a_{1n} = n^2 \left[\frac{\alpha^2 n^2}{12} + K^2 \left(1 + \frac{\alpha^2}{12} \right) \right]$$

$$a_{2n} = \frac{\alpha^2 n^2}{12} \left(2 + \frac{\alpha^2}{4} \right) + K^2 \left(1 + \frac{\alpha^2}{12} \right) \left(1 + \frac{\alpha^2}{4} \right)$$

$$a_{3n} = \frac{\alpha^2}{12} \left(1 - \frac{\alpha^2}{12} \right)$$

$$a_{4n} = n \left[\frac{\alpha^2 n^2}{12} + K^2 \left(1 + \frac{\alpha^2}{12} \right) \right]$$

$$a_{5n} = \frac{\alpha^2 n}{12} \left[1 + 2K^2 + \frac{\alpha^2}{6}\,(1 + K^2) \right]$$

$$a_{6n} = K^2 \left(1 + \frac{\alpha^2}{12} \right) n(n^2 - 1)$$

$$a_{7n} = n \left[K^2 \left(1 + \frac{\alpha^2}{12} \right)^2 + \frac{\alpha^2}{12} \left(2 + \frac{\alpha^2}{12} \right) \right]$$

$K = \sqrt{\frac{1 - \nu}{2}}\ \kappa$ where κ is the Mindlin shear coefficient. $\kappa^2 = 0.86$ in the example of Section IV.

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